PMATH 336 - Practice Problems

The purpose of this document is to provide you with as many exercises as possible for you to get some extra practice. More problems can be found in the texts for the course, and in the exercises I give in class. There may be overlap with questions here and questions in your assignments and quizzes, and maybe even your midterm and final exam! Although no solutions will be provided, I would be happy to discuss any of these problems in office hours.

1 Modular Arithmetic (Math 135 Review)

1. (a) *(Assignment 1)* Let n be a positive integer. Prove that for an integer a, there is an $x \in \mathbb{Z}$ such that $ax = 1$ if and only if $\gcd(a, n) = 1$.

   (b) Complete the following table of inverses in $\mathbb{Z}_{15}$:

x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
x^{-1}	$*$	1	8	$*$											

2 Introduction to Groups

1. *(Assignment 1)* Determine whether or not each of the following sets forms a group under the given operation.

   (a) $(\mathbb{Q}^+, *)$, where $\mathbb{Q}^+ = \{a \in \mathbb{Q} : a > 0\}$ and $a * b = \frac{ab}{2}$

   (b) $(\mathbb{R}, \star)$, where $a \star b = a + b + ab$.

   (c) $G = \left\{ \begin{bmatrix} a & a \\ a & a \end{bmatrix} : a \in \mathbb{R}^* \right\}$ under matrix multiplication.

2. Let G be a set and $\cdot : G \times G \to G$ be a binary operation on G. Suppose that $(G, \cdot)$ satisfies the first two properties of a group (associativity and identity). Prove that if $a, b, c \in G$ are such that $ab = e$ and $ca = e$, then $b = c$.

   This shows that we could change the third axiom of a group to the following and we would still define a group: *For every $a \in G$, there are elements $b, c \in G$ such that $ab = e$ and $ca = e$.*

3. *(Assignment 1)* Let G be a group.

   (a) Let a be an element of G. Show that the inverse of a is unique.

   (b) If $a_1, a_2, \ldots, a_n \in G$, what is $(a_1 a_2 \ldots a_{n-1} a_n)^{-1}$?

4. Let a be an element of a group G. Prove that $(a^{-1})^{-1} = a$.

5. Let a be an element of a group $(G, \cdot)$. Given an integer n, define

$$a^n = \begin{cases} \overbrace{a \cdot a \cdot \cdots \cdot a}^{n \text{ times}} & \text{if } n > 0 \\ e & \text{if } n = 0 \\ \underbrace{a^{-1} \cdot a^{-1} \cdot \cdots \cdot a^{-1}}_{n \text{ times}} & \text{if } n < 0. \end{cases}$$

Prove that $a^m a^n = a^{m+n}$ and $(a^m)^n = a^{mn}$.

6. Let G be a group. Prove that if $(ab)^2 = a^2 b^2$ for all $a, b \in G$, then G is Abelian.

7. *(Assignment 1)* Let $(G, *)$ and $(H, \star)$ be groups. Consider the set

$$G \times H = \{(g, h) : g \in G, h \in H\},$$

and the operation $(g_1, h_2) \cdot (g_2, h_2) = (g_1 * g_2, h_1 \star h_2)$.

   (a) Prove that $(G \times H, \cdot)$ is a group. This group is called the **direct product** of G and H.

   (b) What is $|\mathbb{Z}_2 \times \mathbb{Z}_3|$?

   (c) In general, what is $|G \times H|$ in terms of $|G|$ and $|H|$? No proof is required.

8. *(Assignment 1)* Let g be an element of a group G. Prove that the function $\phi_g : G \to G$ defined by $\phi_g(h) = gh$ is bijective. *(This fact is extremely important!)*

9. *(Assignment 1)* Let n be a positive integer.

   (a) Let a be a positive integer. Prove that there is an $x \in \mathbb{Z}$ such that $ax = 1 \bmod n$ if and only if $\gcd(a, n) = 1$.

   (b) What are the elements of $\mathbb{Z}_9^*$? Determine the inverse of each element.

   (c) What is $|\mathbb{Z}_p^*|$ when p is prime?

10. For any positive integers n and k, define

$$GL_n(\mathbb{Z}_k) = \{A \in \mathbb{M}_n(\mathbb{Z}_k) : \det(A) \in \mathbb{Z}_k^*\}.$$

It can be shown that $GL_n(\mathbb{Z}_k)$ is a group under matrix multiplication (where the operations are performed modulo k). You do not need to show this.

   (a) Does $\begin{bmatrix} 12 & 2 \\ 4 & 1 \end{bmatrix}$ belong to $GL_n(\mathbb{Z}_{15})$? If so, determine its inverse in this group.

   (b) List every element of $GL_2(\mathbb{Z}_2)$. Is this group Abelian?

11. *(Assignment 1)* Let G be a group with identity e. The **order of an element** $a \in G$, denoted $|a|$, is the smallest positive integer n such that $a^n = e$. If no such n exists, we say that $|a| = \infty$.

    Determine the order of each group element.

    (a) The element 3 of $\mathbb{Z}_{14}^*$.

    (b) The element i of the multiplicative group $\{1, i, -1, -i\}$.

    (c) F, a flip in D_n.

    (d) R, a counterclockwise rotation by $2\pi/n$ radians in D_n.

    (e) The matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ in $GL_2(\mathbb{R})$.

12. Determine the order of every element in $\mathbb{Z}_4$. Do the same for $\mathbb{Z}_9^*$ and D_4. Notice anything interesting?

13. *(Assignment 1)* Let G be a group with the property that $a^2 = e$ for all $g \in G$.

    (a) Prove that G is Abelian.

    (b) Find an example of such a group G with $|G| > 2$.

14. *(Assignment 1)* We say that two finite groups G and H are **isomorphic** if, after relabelling and reordering their elements, the Cayley tables for G and H are identical.

    (a) Are $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$? Explain.

    (b) Let G be a finite group. Prove that every group element appears exactly once in every row and column of the Cayley table for G.

    (c) Prove that any two groups of order 3 are isomorphic.

15. (a) What feature of a Cayley table would indicate that a group G is Abelian?

    (b) According to the definition of isomorphic groups given in the problem above, can two finite groups G and H be isomorphic if G is Abelian and H is non-Abelian? Explain.

    (c) Can two finite groups G and H be isomorphic if G has an element of order 2 but H does not? Explain.

    (d) Give examples of non-isomorphic groups G and H of order 6. Find two non-isomorphic groups of order 8.

16. Let $G = \{e, a, b, c, d\}$ be a group. Complete the Cayley table for G:

	e	a	b	c	d
e	e	_	_	_	_
a	_	b	_	_	e
b	_	c	d	e	_
c	_	d	_	a	b
d	_	_	_	_	_

17. Let $G = \{e, a, b, c\}$ be a group of order 4. Write down the possible Cayley tables of G. How many non-isomorphic groups of order 4 exist?

18. *(Assignment 1 Bonus)* Let G be a group with identity e. Suppose that $a^{88} = e$ and $a^{18} = e$ for all $a \in G$. Prove that G is Abelian.

19. *(Assignment 2)* In class we saw that for every $a \in \mathbb{Z}_n$, the order of a divides $n = |\mathbb{Z}_n|$. In this question you will prove that if G is *any* finite Abelian group, then the order of a group element divides $|G|$. Think about whether or not this should hold when G is non-Abelian.

    Let $G = \{g_1, g_2, \ldots, g_n\}$ be an Abelian group of order $n < \infty$. Fix an element $a \in G$.

    (a) Prove that $g_1 g_2 \cdots g_n = (ag_1)(ag_2) \cdots (ag_n)$.

    (b) Deduce that $|a|$ divides $|G|$.

    (c) Prove **Euler's Theorem**: If a and n are integers with $\gcd(a, n) = 1$, then
    $$a^{\varphi(n)} = 1 \mod n.$$
    Here, φ denotes the Euler totient function as in problem 7.

20. Given an example of an infinite Abelian group with exactly two elements of order 4.

3 Dihedral Groups

1. Consider the group D_3 of symmetries of an equilateral triangle.

    (a) *(Assignment 1)* Describe the elements of D_3 and write down the group's Cayley table.

    (b) Let $R \in D_3$ be a counterclockwise rotation by $2\pi/3$ radians, and let $F \in D_3$ be any flip. Show explicitly that every symmetry from (a) can be written as R^k or FR^k for some $k \geq 0$.

2. What is $|D_n|$?

3. Let $n \geq 3$. Let $R \in D_n$ be a rotation and $F \in D_n$ be a flip.

    (a) Prove that for any integer k, FR^k is a flip. Deduce that $FR^kF = R^{-k}$.

    (b) Conclude that D_n is non-Abelian.

4. Compute the order of every element of D_4.

4 Symmetric Groups

1. Prove that S_n is a group for all $n \geq 1$.

2. Consider the elements $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 3 & 2 & 5 \end{pmatrix}$ and $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 2 & 1 & 4 \end{pmatrix}$. Compute $\tau\sigma$, $\sigma\tau$, σ^2, τ^2, σ^{-1} and τ^{-1}.

3. Write down the Cayley table for S_3. Compare this with the Cayley table for D_3. Are these groups isomorphic?

4. *(Assignment 1)* For each $k \in \{1, 2, 3, 4, 5\}$, find a permutation $\pi \in S_5$ such that $|\pi| = k$.

5. (a) Explain how one can view D_n as a subgroup of S_n.

   (b) Based on your answer in (a), write down the elements of D_4 as products of disjoint cycles in S_4.

6. Write each permutation as a product of disjoint cycles

   (a) $(1234)(2543)$

   (b) $(12)(12345)(12)$

   (c) $(12)(13)(24)(23)(153)$

   (d) $(15423)(25143)(354)(1234)$

7. For each permutation σ, find $|\sigma|$ and σ^{-1}.

   (a) $\sigma = (12467)$

   (b) $\sigma = (1, 2)(3, 4, 5)(6, 7, 8, 9, 10)(11, 12, 13)$

   (c) $\sigma = (12 \cdots n)(12 \cdots n)$

   (d) $\sigma = (12)(123)(1234)$

8. Determine whether each permutation is even or odd.

   (a) (12345)

   (b) $(1345)(142)$

   (c) $(12)(123)(1234)$

   (d) $(12)(123)(1234) \cdots (12 \cdots n)$.

   (e) $(12 \cdots m)(12 \cdots n)$.

9. What are the possible orders of elements in S_6?

10. Determine the number of elements of order 5 in S_7

11. Determine the number of elements of order 10 in S_9

12. *(Assignment 3)*

    (a) Determine the number of elements of order 4 in S_6.

    (b) How many elements in A_6 have order 4? Explain.

13. Show that the equation $x^2 = (1234)$ has no solution in S_7.

14. Determine whether each permutation is even or odd.

    (a) (1234)

    (b) $(123)(45678)$

    (c) $(12)(123)(1234)$

    (d) $(12)(123)(1234)\cdots(1\cdots n)$

    (e) $(1523)(14375)(237)(1365)$

    (f) $(1\cdots n)(1\cdots m)$

15. Prove that an m-cycle in S_n is even if and only if m is odd.

16. Let σ be an m-cycle in S_n. Prove that if m is odd, then σ^2 is also an m-cycle.

17. Prove that if $\sigma \in S_n$ has odd order, then σ is an even permutation.

18. *(Assignment 3)* For each positive integer $n \geq 3$, let A_n denote the subset of S_n consisting of all even permutations.

    (a) Prove that A_n is a subgroup of S_n. We call A_n the *alternating group*.

    (b) Find a bijection between A_n and $S_n \setminus A_n$ (the set of odd permutations in S_n). Deduce that $|A_n| = n!/2$.

    (c) Write all 12 elements of A_4 in cycle notation.

    (d) Show that A_n is non-Abelian for all $n \geq 4$.

19. Prove that if H is a subgroup of S_n, then either every member of H is even or exactly half of the members of H are even.

20. (a) Watch *Futurama* Season 10, Episode 6: *The Prisoner of Benda* (if you're not sure where to find this episode, maybe Google knows?)

    (b) Solve the problem in this episode by proving the following fact: If σ is a permutation in S_n, then σ can be written as a product of distinct transpositions in S_{n+2}, each involving at least one of the elements $n+1$ or $n+2$.

5 Subgroups

1. This question concerns the intersection of subgroups.

   (a) Let H and K be subgroups of a group G. Prove that $H \cap K$ is a subgroup of G.

   (b) More generally, prove that if H_i, $i \in \mathcal{I}$ is an arbitrary collection of subgroups of G, then $\bigcap_{i \in \mathcal{I}} H_i$ is a subgroup of G.

2. Let a be an element of a group G. The **centralizer** of a is the set

$$C(a) = \{b \in G : ab = ba\}.$$

   Prove that $C(a) \leq G$.

3. Let G be a group. Define the **centre** of G to be the set

$$Z(G) = \{a \in G : ab = ba \text{ for all } b \in G\}.$$

   (a) Prove that $Z(G) \leq G$.

   (b) Determine the centre of each of the following groups:

      i. $\mathbb{Z}_n$

      ii. D_3

      iii. D_4

      iv. S_3

      v. S_4

4. Let a be an element of a group G. Prove that $\langle a \rangle$ is a subgroup of G. Prove that $|a| = |\langle a \rangle|$.

5. Let a be an element of a group G. Prove that $\langle a \rangle$ is the smallest subgroup of G containing a. That is, prove that if H is a subgroup of G that contains a, then $\langle a \rangle \subseteq H$.

6. Let G be a group. Prove that if $H \leq G$ and $K \leq H$, then $K \leq G$.

7. *(Assignment 2)* Let G be a group. Prove the **one-step subgroup test**: A non-empty subset H of G is a subgroup of G if and only if $ab^{-1} \in H$ for all $a, b \in H$.

8. Prove the **finite subgroup test**: Given a *finite* group G, a non-empty subset H of G is a subgroup of G if and only if H is closed under the group operation. (i.e., you don't need to check inverses!)

9. Prove of disprove:

(a) If G is a finite group, then every element of G has finite order.

(b) If G is an infinite group, then G has at least one element of G has infinite order.

10. *(Assignment 2)* We know from linear algebra that a rotation in $\mathbb{R}^2$ can be expressed as a matrix
$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \text{ for some } \theta \in \mathbb{R}.$$
Prove that $G = \left\{ \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} : \theta \in \mathbb{R} \right\}$ is a subgroup of $SL_2(\mathbb{R})$. This group is the rotational symmetry group of a circle.

11. Let G be a finite group of order $n > 2$. Prove that G does not contain a subgroup of order $n-1$.

12. Let a and b be elements of a group. Prove that $|ab| = |ba|$.

13. Prove or disprove: For all group elements a and b, $|ab| = |a||b|$.

14. Let a be an element of a group G, let b be an element of a group H, and assume that $|a|, |b| < \infty$. Prove that the order of $(a, b) \in G \times H$ is $\operatorname{lcm}(|a|, |b|)$.

15. Draw the subgroup lattice for $\mathbb{Z}_{30}$

16. Draw the subgroup lattice for $\mathbb{Z}_{18}^*$.

17. Draw the subgroup lattice for $\mathbb{Z}_{pq}$ where p and q are distinct primes.

18. Draw the subgroup lattice for $\mathbb{Z}$. Note that this lattice will be infinite, so start at the top and draw as much as you can.

6 Cyclic Groups

1. Determine whether or not each of the following groups is cyclic. If it is cyclic, find all generators of the group.

   (a) $\mathbb{Z}_{12}$

   (b) $\mathbb{Z}_p$ where p is prime

   (c) D_3

   (d) $\mathbb{Z}_{25}^*$

   (e) S_4

   (f) $\{\pm 1, \pm i\}$

(g) The subgroup $\left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} : n \in \mathbb{Z} \right\}$ of $GL_2(\mathbb{R})$.

2. Find a cyclic subgroup of D_n of order n.

3. We know that any subgroup of a cyclic group is cyclic. Furthermore, we know from one of the questions above that the intersection of two subgroups is again a subgroup. With these facts in mind...

   (a) Find a generator for $6\mathbb{Z} \cap 15\mathbb{Z}$, the intersection of two subgroups of the cyclic group $\mathbb{Z}$

   (b) More generally, find a generator for $n\mathbb{Z} \cap m\mathbb{Z}$.

4. *(Assignment 2)*

   (a) Let n be a positive integer. Prove that the set

$$G = \{z \in \mathbb{C} : z^n = 1\}$$

   of all n^{th} roots of unity is a cyclic subgroup of $\mathbb{C}^*$. What is $|G|$?

   (b) Let G be the group of 12^{th} roots of unity.

      (ii) Find all generators of G.

      (iii) Find all subgroups of G.

5. (a) How many subgroups does $\mathbb{Z}_{10}$ have?

   (b) How many subgroups does $\mathbb{Z}_{600}$ have?

6. What are the subgroups of $\mathbb{Z}_{60}$?

7. Find all elements of $\mathbb{Z}_{42}$ with order 6.

8. Let a and b be elements of a group. If $|a| = 10$ and $|b| = 21$, show that $\langle a \rangle \cap \langle b \rangle = \{e\}$.

9. Let a and b be elements of a group. If $|a| = 48$ and $|b| = 72$, what are the possibilities for $|\langle a \rangle \cap \langle b \rangle|$?

10. Let a be an element of a group. If $|a^{28}| = 10$ and $|a^{22}| = 20$, determine $|a|$.

11. *(Assignment 2)*

   (a) Find the order of 85 in $\mathbb{Z}_{105}$.

   (b) Let $G = \langle a \rangle$ be a cyclic group of order 72. Find all elements of G of order 18.

   (c) If p is prime, what are the subgroups of $\mathbb{Z}_{p^2}$? What about $\mathbb{Z}_{p^k}$ for $k \in \mathbb{N}$?

12. *(Assignment 2)* This question concerns cyclic groups and direct products.

    (a) Let G and H be groups with $|G| \geq 2$ and $|H| \geq 2$. Prove that if $G \times H$ is cyclic, then G is cyclic and finite (the same is true of H but you don't need to prove it!)

    (b) Show that $\mathbb{Z}_4 \times \mathbb{Z}_5$ is cyclic. What are the generators of this group?

    (c) Show that $\mathbb{Z}_4 \times \mathbb{Z}_6$ is not cyclic.

        (Hmm... This demonstrates that a direct product of finite cyclic groups may or may not be cyclic. Think about what conditions one could put on finite cyclic groups G and H to guarantee that $G \times H$ is cyclic.)

13. *(Assignment 2)*

    (a) Prove or disprove that $(\mathbb{Q}, +)$ cyclic.

    (b) Prove or disprove that $(\mathbb{R}, +)$ cyclic.

14. *(Assignment 2)* In this question you will use group theory to prove an important fact about the Euler totient function φ. Recall that for a positive integer n, we define

$$\varphi(n) = |\{1 \leq a \leq n : \gcd(a, n) = 1\}|.$$

    i.e., $\varphi(n)$ is the number of positive integers less than n that are coprime to n.

    (a) Prove that for each integer $n \geq 3$, $\mathbb{Z}_n$ has an even number of generators.

    (b) Conclude that $\varphi(n)$ is even for $n > 2$. What does this tell you about $\mathbb{Z}_n^*$?

15. Let G be a group. Prove that if p is a prime and G has more than $p - 1$ elements of order p, then G is not cyclic.

16. Let $G = \langle a \rangle$ be a cyclic group of order 100. Draw the subgroup lattice for G.

7 Cosets and Lagrange's Theorem

1. Find all left cosets of $\{0, 5\}$ in $\mathbb{Z}_{10}$

2. Find all left cosets of $\{1, 11\}$ in $\mathbb{Z}_{20}^*$.

3. Find all left cosets of $\{e\}$ in D_4

4. Consider $\mathbb{Z}$ as a subgroup of $\mathbb{Q}$. What would it mean for $a, b \in \mathbb{Q}$ to satisfy $a+\mathbb{Z} = b+\mathbb{Z}$?

5. *(Assignment 3)* Find all left cosets of $5\mathbb{Z}$ in $\mathbb{Z}$.

6. *(Assignment 3)*

(a) Let V be the vertical flip in D_4. Find all left cosets of $\langle V \rangle$ in D_4, and find all right cosets of $\langle V \rangle$ in D_4. Notice that there are the same number of each.

(b) Find an element $a \in D_4$ such that $a\langle V \rangle \neq \langle V \rangle a$.

7. *(Assignment 3)* Consider the subgroup $SL_n(\mathbb{R})$ of $GL_n(\mathbb{R})$. Find a condition on $A, B \in GL_n(\mathbb{R})$ that characterizes when $A \cdot SL_n(\mathbb{R}) = B \cdot SL_n(\mathbb{R})$.

8. The set $\mathbb{M}_n(\mathbb{C})$ of all $n \times n$ complex matrices forms a group under matrix addition. The set
$$\mathcal{S} = \{A \in \mathbb{M}_n(\mathbb{C}) : \mathrm{Tr}(A) = 0\}$$
is a subgroup of $\mathbb{M}_n(\mathbb{C})$ (recall from linear algebra that $\mathrm{Tr}(A)$ denotes the trace A, the sum of A's diagonal entries). Find a condition on $A, B \in \mathbb{M}_n(\mathbb{C})$ that characterizes when $A + \mathcal{S} = B + \mathcal{S}$.

9. The set $\mathcal{F} = \{f : \mathbb{R} \to \mathbb{R} \,|\, f \text{ is bijective}\}$ is a group under function composition. The set
$$\mathcal{S} = \{f \in \mathcal{F} : f(0) = 0\}$$
is a subgroup of $\mathcal{F}$. Find a condition on $f, g \in \mathcal{F}$ that characterizes when $f \circ \mathcal{S} = g \circ \mathcal{S}$.

10. Let H be a subgroup of G. Prove that for $a, b \in G$, either $aH = bH$ or $aH \cap bH = \emptyset$.

11. Let H be a subgroup of G. Prove that $|aH| = |H|$ for all $a \in G$.

12. Prove that if G is a finite group and $H \leq G$, then $|G : H| = |G|/|H|$.

13. Let G be a finite group, $H \leq G$, and $K \leq G$. Prove that $|G : K| = |G : H| \cdot |H : K|$.

14. *(Assignment 3)* Let a be a group element of order 30. Determine the number of left cosets of $\langle a^4 \rangle$ in $\langle a \rangle$. Write down the cosets.

15. Prove that if a is an element of a finite group G, then $|a|$ divides $|G|$.

16. Prove that if p is a prime and G is a group of order p, then G is cyclic.

17. Prove that if a is an element of a finite group G, then $a^{|G|} = e$.

18. Prove **Fermat's Little Theorem**: If p is a prime and a is a positive integer such that p does not divide a, then $a^{p-1} = 1 \bmod p$.

19. *(Assignment 3)* Let p and q be primes and let G be a group of order pq. Prove that every proper subgroup of G is cyclic.

20. Show that the converse to Lagrange's Theorem is false. That is, find a finite group G and positive divisor n of $|G|$ such that G has no subgroup of order n.

21. Let G be a group of order 155. Suppose that $a, b \in G$ are non-identity elements of different orders. Prove that the only subgroup of G containing a and b is G itself.

22. Let H and K be subgroups of a group G. If $|H| = 22$ and $|K| = 35$, prove that $H \cap K = \{e\}$.

23. *(Assignment 3 Bonus)* Let $m_1, m_2, \ldots, m_k$ be positive integers and define $n = m_1 + m_2 + \cdots + m_k$. Prove that $(m_1!)(m_2!)\cdots(m_k!)$ divides $n!$.

24. Let m and n be positive integers. Prove that $(m!)^n(n!)$ divides $(mn)!$.

25. *(Assignment 3)* Prove that if $H \leq S_n$ and $|H|$ is odd, then $H \leq A_n$.

26. *(Assignment 3)* Let $G = \{a_1, a_2, \ldots, a_n\}$ be an Abelian group of order n. Prove that if n is odd, then $a_1 a_2 \cdots a_n = e$.

27. On Assignment 2 you proved that $|\mathbb{Z}_n^*|$ is even for all $n > 2$. Prove this fact in one line using the following corollary to Lagrange's theorem: for all $a \in G$, $|a|$ divides $|G|$.

8 Quotient Groups and Normal Subgroups

1. Prove that every subgroup of an Abelian group is normal.

2. Prove that for all $n \geq 3$, $A_n \trianglelefteq S_n$.

3. Let H be the subgroup of rotations in D_n. Prove that $H \trianglelefteq D_n$.

4. Let G be a group. Prove that the centre $Z(G)$ is a normal subgroup of G.

5. *(Assignment 4)* Prove that $SL_n(\mathbb{R}) \trianglelefteq GL_n(\mathbb{R})$.

6. Prove that the subgroup $\langle(12)\rangle = \{e, (12)\}$ of S_3 is not normal.

7. Let G be a group and H be a subgroup of G of order n. Prove that if H is the only subgroup of G with order n, then $H \trianglelefteq G$.

8. Let G be a group and H be a subgroup of G of order n. Prove that the intersection of all subgroups of G of order n is a normal subgroup of G.

9. *(Assignment 4)* Let G be a group and $H \trianglelefteq G$. Prove that if $|G : H| = n < \infty$, then $a^n \in H$ for all $a \in G$.

10. A group G is called **simple** if the only normal subgroups of G are $\{e\}$ and G.

    (a) Determine all finite simple Abelian groups.

(b) Let G and H be groups with $|G| \geq 2$ and $|H| \geq 2$. Prove that $G \times H$ is not simple.

(c) Prove that D_n is not simple for all $n \geq 3$.

(d) Prove that A_4 is not simple.

11. *(Assignment 4)* This question is about the group Q_8, known as the **quaternions**. As a set,

$$Q_8 = \{1, -1, i, -i, j, -j, k, -k\}.$$

We can define a group operation on Q_8 using the relations

$$i^2 = j^2 = k^2 = -1,$$
$$1^2 = (-1)^2 = 1,$$
$$(-1)i = -i, \quad (-1)j = -j, \quad (-1)k = -k,$$
$$ij = k, \quad jk = i, \quad ki = j.$$

This operation turns Q_8 into a group—you don't need to prove this.

(a) Using the relations above, write down the Cayley table for Q_8.

(b) Determine all subgroups of Q_8. Which of these subgroups is $Z(Q_8)$?

(c) Prove that although Q_8 is non-Abelian, every subgroup of Q_8 is normal.

12. Let G be a group and consider the group $G \times G$. We define the **diagonal** of $G \times G$ as the set

$$\Delta_G = \{(a, a) : a \in G\}.$$

One can show that $\Delta_G \leq G \times G$. Prove that G is Abelian if and only if Δ_G is normal.

13. Let G be a group of order $2p$ where p is prime. Prove that if $H \leq G$ and H is not normal, then $|H| = 2$.

14. *(Assignment 4)* Let G be a group and $H, K \leq G$. Prove that if $H \trianglelefteq G$ then $HK \leq G$.

15. Prove that if G is an Abelian group and $H \trianglelefteq G$, then G/H is Abelian. Is the converse true?

16. Prove that if G is a cyclic group and $H \trianglelefteq G$, then G/H is cyclic. Is the converse true?

17. What is the order of $14 + \langle 8 \rangle$ in the quotient group $\mathbb{Z}_{24}/\langle 8 \rangle$?

18. Let G be a group and $H \trianglelefteq G$. Suppose that $|G| = 48$, $|H| = 3$, and $G/H = \langle aH \rangle$ is cyclic. What is the order of $(aH)^6$?

19. Recall that $Z(D_4) = \{e, R_{180}\}$. Write down the Cayley table for $D_4/Z(D_4)$.

20. Let G be a group and $N \trianglelefteq G$. Let H be a subgroup of G such that $N \subseteq H$.

    (a) Prove that $N \trianglelefteq H$.

    (b) Prove that H/N is a subgroup of G/N.

    (c) Prove that $H/N \trianglelefteq G/N$ if and only if $H \trianglelefteq G$.

21. Prove that if G is a group of order pq where p and q are primes, then either G is Abelian or $Z(G) = \{e\}$.

22. *(Assignment 4)* Find an example of a group G such that $|G : Z(G)|$ is prime, or show that no such group exists.

23. Prove that an Abelian group of order 33 is cyclic.

24. *(Assignment 4)* Prove that if G is an Abelian group of order pq where p and q are distinct primes, then G is cyclic.

9 Homomorphisms and Isomorphisms

1. We have seen two definitions of isomorphisms:

    (i) (Assignment 1) Two finite groups G and H are **isomorphic** if, after relabelling and reordering their elements, the Cayley tables for G and H are identical.

    (ii) (Section 7 from class) Two groups G and H are **isomorphic** if there is a bijective homomorphism $\varphi : G \to H$.

    Compare these two definitions. Explain why they are equivalent (at least for finite groups!)

2. Let G, H, and K be group. Prove that $\cong$ is an **equivalence relation**. That is,

    (a) Prove that $G \cong G$.

    (b) Prove that if $G \cong H$, then $H \cong G$.

    (c) Prove that if $G \cong H$ and $H \cong K$, then $G \cong K$.

3. Let $\varphi : G \to H$ be a group homomorphism.

    (a) Prove that $\varphi(e) = e'$, where e and e' are the identities for G and H, respectively.

    (b) Let $a \in G$. Prove that $\varphi(a^{-1}) = \varphi(a)^{-1}$.

    (c) Let $a \in G$. Prove that $|\varphi(a)|$ divides $|a|$.

4. Let $\varphi : G \to H$ and $\psi : H \to K$ be group homomorphisms. Prove that $\psi \circ \varphi : G \to K$ is a homomorphism.

5. *(Assignment 4)* Recall from a previous problem that $SL_n(\mathbb{R})$ is a normal subgroup of $GL_n(\mathbb{R})$. Construct an isomorphism $\varphi : GL_n(\mathbb{R})/SL_n(\mathbb{R}) \to \mathbb{R}^*$. Justify that φ is indeed an isomorphism.

6. *(Assignment 4)* Prove that $\mathbb{Z}_m \times \mathbb{Z}_n \cong \mathbb{Z}_{mn}$ if and only if $\gcd(m, n) = 1$.

7. Determine, with justification, whether or not the following groups are isomorphic.

   (a) $(\mathbb{Z}, +)$ and $(\mathbb{Z} \times \mathbb{Z}, +)$.

   (b) $(\mathbb{Z} \times \mathbb{Z}, +)$ and $(\mathbb{Q}, +)$.

   (c) $(\mathbb{Q}, +)$ and $(\mathbb{Q}^*, \cdot)$.

   (d) $(\mathbb{R}^*, \cdot)$ and $(\mathbb{C}^*, \cdot)$.

   (e) $GL_2(\mathbb{R})$ and $\left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a, b \in \mathbb{R}^* \right\}$ under matrix multiplication.

   (f) $(\mathbb{Z}, +)$ and $\left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} : n \in \mathbb{Z} \right\}$ under matrix multiplication.

   (g) $\mathbb{Z}_7^*$ and $\mathbb{Z}_{14}^*$.

   (h) $\mathbb{Z}_2 \times \mathbb{Z}_5$ and $\mathbb{Z}_{10}$.

   (i) $\mathbb{Z}_2 \times \mathbb{Z}_4$ and $\mathbb{Z}_8$.

   (j) Q_8 and D_4.

   (k) S_3 and D_3.

   (l) S_4 and D_{12}.

8. *(Assignment 4)*

   (a) If $\varphi : \mathbb{Z}_{15} \to \mathbb{Z}_5$ is a homomorphism with $\varphi(1) = 3$, what is $\varphi(12)$?

   (b) If $\varphi : S_3 \to S_3$ is a homomorphism with $\varphi((123)) = e$ and $\varphi((12)) = (23)$, what is $\varphi((13))$?

   (c) If $\varphi : \mathbb{Z}_4 \times \mathbb{Z}_3 \to \mathbb{Z}_{24}$ is a homomorphism with $\varphi((1, 0)) = 6$ and $\varphi((0, 1)) = 16$, find $\varphi((1, 1))$ and $\varphi((3, 2))$.

   (d) What are the kernels and images of the homomorphisms above?

9. *(Assignment 4)* Prove that if p and q are distinct primes, then there is only one homomorphism $\varphi : \mathbb{Z}_p \to \mathbb{Z}_q$.

10. *(Assignment 4)* Determine the number of surjective homomorphisms $\varphi : \mathbb{Z}_{20} \to \mathbb{Z}_{10}$.

11. Let G be a group and $a \in G$.

    (a) Consider the function $\varphi : G \to G$ given by $\varphi(b) = ab$. Is φ an isomorphism? Explain.

    (b) Consider the function $\varphi : G \to G$ given by $\varphi(b) = aba^{-1}$. Is φ an isomorphism? Explain.

12. Consider the function $\text{sgn} : S_n \to \mathbb{Z}_2$ given by

$$\text{sgn}(\sigma) = \begin{cases} 0 & \text{if } \sigma \text{ is even} \\ 1 & \text{if } \sigma \text{ is odd} \end{cases}.$$

    Prove that sgn is a homomorphism. What is its kernel?

13. Let $\varphi : G \to H$ be a group homomorphism.

    (a) Prove that $\text{im}(\varphi)$ is a subgroup of H.

    (b) Prove that $\ker(\varphi)$ is a normal subgroup of G.

    (c) Find an example of a group homomorphism $\varphi : G \to H$ such that $\text{im}(\varphi)$ is *not* normal in H.

14. Prove the **First Isomorphism Theorem:** If G, H are groups and $\varphi : G \to H$ is a group homomorphism, then $G/\ker(\varphi) \cong \text{im}(\varphi)$.

15. Let G and H be finite groups and $\varphi : G \to H$ a group homomorphism. Prove that $|\ker(\varphi)|$ and $|\text{im}(\varphi)|$ divide $|G|$.

16. Let $\varphi : G \to H$ be a surjective group homomorphism. If $|G| = 20$ and $|H| = 4$, find $|\ker(\varphi)|$.

17. Let $\varphi : G \to H$ be a group homomorphism. If $|H| = 60$ and $|H : \text{im}(\varphi)| = 12$, determine $|G : \ker(\varphi)|$.

18. Let φ be a surjective group homomorphism from $\mathbb{Z}_{30}$ to a group of order 6. Determine $\ker(\varphi)$.

19. Use the First Isomorphism Theorem to show that $GL_n(\mathbb{R})/SL_n(\mathbb{R}) \cong \mathbb{R}^*$.

20. Prove that $S_n/A_n \cong \mathbb{Z}_2$. Conclude that $|S_n : A_n| = 2$.

21. In this question we will use the symbol $|\cdot|$ to denote the modulus (or absolute value) of a complex number: if $z = a + ib \in \mathbb{C}$, then $|z| = \sqrt{a^2 + b^2}$. Consider the set $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

    (a) Prove that $\mathbb{T}$ is a normal subgroup of $\mathbb{C}^*$.

(b) Prove that $\mathbb{C}^*/\mathbb{T} \cong \mathbb{R}_{>0}$, where $\mathbb{R}_{>0} = \{x \in \mathbb{R} : x > 0\}$ is a group under multiplication.

22. Let G_1 and G_2 be groups, and suppose that $N_1 \trianglelefteq G_1$ and $N_2 \trianglelefteq G_2$.

    (a) Prove that $N_1 \times N_2 \trianglelefteq G_1 \times G_2$.

    (b) Prove that $(G_1 \times G_2)/(N_1 \times N_2) \cong (G_1/N_1) \times (G_2/N_2)$.

23. *(Assignment 5)* Let G be a group and $N \trianglelefteq G$. By the correspondence theorem, every subgroup of G/N is of the form H/N, where $H \leq G$ and $N \subseteq H$. Moreover, by a previous practice problem we know that $H/N \trianglelefteq G/N$ if and only if $H \trianglelefteq G$.

    Let H be a normal subgroup of G with $N \subseteq H$. Prove that $(G/N)/(H/N) \cong G/H$. This result is the **Third Isomorphism Theorem**.

24. *(Assignment 5)* Consider the groups $\mathbb{R}$ and $\mathbb{Z}$ under addition, as well as the subgroup

$$\mathbb{T} = \{z \in \mathbb{C}^* : |z| = 1\}$$

    of $\mathbb{C}^*$—you don't need to prove that $\mathbb{T}$ is a subgroup. Prove that $\mathbb{R}/\mathbb{Z} \cong \mathbb{T}$.

25. *(Assignment 5)* If $\varphi : \mathbb{Z}_{15} \to S_3 \times S_3$ is a non-trivial homomorphism, what is $\ker(\varphi)$? Explain.

26. *(Assignment 5)* Let G be a group and $N \trianglelefteq G$. Complete the proof of the correspondence theorem by showing that every $K \leq G/N$ can be written as $K = H/N$ for some subgroup $H \leq G$ with $N \subseteq H$.

27. *(Assignment 5)* Use the correspondence theorem to find all subgroups of $D_4/Z(D_4)$.

28. Let $\mathbb{Z}[x]$ be the set of all polynomials with coefficients in $\mathbb{Z}$. It can be shown that $\mathbb{Z}[x]$ is a group under polynomial addition. Consider the derivative map $\frac{d}{dx} : \mathbb{Z}[x] \to \mathbb{Z}[x]$ given by

$$\frac{d}{dx}\left(a_0 + a_1x + a_2x^2 + \cdots + a_mx^m\right) = a_1 + 2a_2x + 3a_3x^2 + \cdots + ma_mx^{m-1}.$$

    (a) Show that $\frac{d}{dx}$ is a homomorphism.

    (b) Find the kernel and image of $\frac{d}{dx}$.

29. Let $\mathbb{R}[x]$ be the set of all polynomials with coefficients in $\mathbb{R}$. It can be shown that $\mathbb{R}[x]$ is a group under polynomial addition. Consider the derivative map $\frac{d}{dx} : \mathbb{R}[x] \to \mathbb{R}[x]$ given by

$$\frac{d}{dx}\left(a_0 + a_1x + a_2x^2 + \cdots + a_mx^m\right) = a_1 + 2a_2x + 3a_3x^2 + \cdots + ma_mx^{m-1},$$

and the anti-derivative map $\int : \mathbb{R}[x] \to \mathbb{R}[x]$ given by

$$\int \left(a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m\right) = a_0 x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \cdots + \frac{a_m}{m+1}x^{m+1}$$

(in this case we define $\int$ so that the "$+C$" term is 0).

(a) Show that $\frac{d}{dx}$ and $\int$ are homomorphisms.

(b) Find the kernel and image of $\frac{d}{dx}$, and the kernel and image of $\int$.

30. Let p be a prime and consider the set $\mathbb{Z}_p[x]$ of all polynomials with coefficients in $\mathbb{Z}_p$. It can be shown that $\mathbb{Z}_p[x]$ is a group under polynomial addition. Consider the derivative map $\frac{d}{dx} : \mathbb{Z}_p[x] \to \mathbb{Z}_p[x]$ given by

$$\frac{d}{dx}\left(a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m\right) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + m a_m x^{m-1}$$

where the coefficients are reduced mod p.

(a) Show that $\frac{d}{dx}$ is a homomorphism.

(b) Find the kernel and image of $\frac{d}{dx}$.

10 Automorphisms

1. Let G be a group. Prove that $\mathrm{Aut}(G)$ is a group under function composition.

2. Let G be a group. Prove that $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$.

3. Let G be a group. Prove that the map $\varphi : G \to G$ given by $\varphi(a) = a^{-1}$ is an automorphism of G if and only if G is Abelian.

4. Determine $\mathrm{Aut}(\mathbb{Z})$.

5. Prove that $\mathrm{Aut}(\mathbb{Z}_n) \cong \mathbb{Z}_n^*$.

6. Let G be a group. Show that $\mathrm{Aut}(G \times G)$ contains an automorphism of order 2.

7. Let G be a group. Prove that there is a subgroup of $\mathrm{Aut}(G \times G \times G)$ that is isomorphic to S_3.

8. (a) Prove **Eduardo's Theorem**: if $G \cong H$, then $\mathrm{Aut}(G) \cong \mathrm{Aut}(H)$.

   (b) Show that the converse to part (a) is false.

9. *(Assignment 5)* Let G be a group. Prove that $G/Z(G) \cong \mathrm{Inn}(G)$.

10. *(Assignment 5)* Let G be a group. Prove that if $\mathrm{Aut}(G)$ is cyclic, then G is Abelian.

11. *(Assignment 5 Bonus)* Let $n \geq 3$ be an odd integer. Prove that there is no finite group G such that $\mathrm{Aut(G)} \cong \mathbb{Z}_n$.

12. Let G be a group with $Z(G) = \{e\}$. Prove that $Z(\mathrm{Aut}(G)) = \{\mathrm{id}\}$.

    Hint: What would it mean for an automorphism φ to commute with an inner automorphism φ_g?

11 Group Actions

1. Let X be a non-empty set, and define
$$S_X = \{f : X \to X \mid f \text{ is bijective}\}.$$
   (a) Prove that S_X is a group under function composition.

   (b) Prove that if $|X| = n < \infty$, then $S_X \cong S_n$.

2. Let $G \curvearrowright X$ be a group action.

   (a) Prove that for every $a \in G$, the map $\psi_a : X \to X$ given by $\psi_a(x) = a \cdot x$ is a bijection. That is, prove that $\psi_a \in S_X$.

   (b) Prove that the map $\Phi : G \to S_X$ given by $\Phi(a) = \psi_a$ is a group homomorphism.

3. Let $G \curvearrowright X$ be a group action. Prove that for each $a \in G$, $\mathrm{Stab}_a \leq G$.

4. Let G be a group.

   (a) Prove that the map $a \cdot b = ab$ is an action of G on itself.

   (b) Prove that the map $a \cdot b = aba^{-1}$ is an action of G on itself.

5. The quaternion group Q_8 acts on itself by conjugation. Find the orbit of i, $\mathcal{O}_i$, under this action.

6. Consider the subgroup
$$G = \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} : a, b \in \mathbb{R}, a \neq 0 \right\}$$
   of $GL_2(\mathbb{R})$. Define a (potential) action of G on $X = \mathbb{R}$ by $\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \cdot x = ax + b$.

   (a) Prove that this is indeed an action.

   (b) Determine Stab_0 and $\mathcal{O}_0$.

7. Let $G \curvearrowright X$ be a group action. Prove the **Orbit-Stabilizer Theorem:** For every $x \in X$, $|G : \mathrm{Stab}_x| = |\mathcal{O}_x|$.

8. It is well-known that an $n \times n$ matrix A with integer entries has an inverse with integer entries if and only if $\det(A) = \pm 1$ (this is a fun exercise in linear algebra—try it!). With this in mind, the set

$$GL_n(\mathbb{Z}) := \{A \in \mathbb{M}_n(\mathbb{Z}) : \det(A) = \pm 1\}$$

forms a group under matrix multiplication. Define a (potential) action of $GL_2(\mathbb{Z})$ on $X = \mathbb{Z}^2$ by matrix-vector multiplication: $A \cdot x = Ax$.

   (a) Prove that this is indeed an action.

   (b) Given a vector $x = \begin{bmatrix} m \\ n \end{bmatrix} \in \mathbb{Z}^2$, prove that

$$\mathcal{O}_x = \left\{ \begin{bmatrix} p \\ q \end{bmatrix} \in \mathbb{Z}^2 : \gcd(p,q) = \gcd(m,n) \right\}.$$

9. Let G be a group.

   (a) Prove **Cayley's Theorem:** Every group G is isomorphic to a subgroup of S_X for some set X.

   (b) Deduce that if $|G| = n < \infty$, then G is isomorphic to a subgroup of S_n.

10. *(Assignment 5)* Let G be a group acting on a set X. Prove that the orbits of this action partition X. That is, prove that

   (a) $X = \bigcup_{x \in X} \mathcal{O}_x$, and

   (b) for any $x, y \in X$, either $\mathcal{O}_x = \mathcal{O}_y$ or $\mathcal{O}_x \cap \mathcal{O}_y = \emptyset$.

11. *(Assignment 5)*

   (a) Determine $\mathrm{Aut}(\mathbb{Z}_8)$.

   (b) Let $G = \mathrm{Aut}(\mathbb{Z}_8)$ and $X = \mathbb{Z}_8$. Define an action $G \curvearrowright X$ by $\varphi \cdot a = \varphi(a)$. For each element $a \in \mathbb{Z}_8$, compute $\mathcal{O}_a$ and Stab_a.

12. *(Referenced on Assignment 5)* Let G be a group and $H \leq G$.

   (a) Prove that $a \cdot (gH) = agH$ defines an action of G on the set $X = \{gH : g \in G\}$.

   (b) Consider the element $H \in X$. Determine $\mathcal{O}_H$ and Stab_H.

   (c) Suppose now that G is finite. Becky says that she has a new proof of Lagrange's Theorem using the action above. "Use the Orbit-Stabilizer theorem," she snarls. "Since $|G| = |\mathcal{O}_H| \cdot |\mathrm{Stab}_H|$, the result is immediate from (b)."

   You would like to believe Becky, but you're a bit skeptical; she's lied to you in the past, and frankly you have some trust issues.

   Explain why Becky's argument does **not** give an alternate proof of Lagrange's theorem.

13. *(Assignment 5)* Let G be a finite group, and let H and K be subgroups of G. Consider the set $X = HK = \{hk : h \in H, k \in K\}$.

    (a) Prove that $(h, k) \cdot x = hxk^{-1}$ is an action of $H \times K$ on X.

    (b) Use the Orbit-Stabilizer Theorem to prove that $|HK| = \dfrac{|H||K|}{|H \cap K|}$.

14. *(Assignment 5)* In this problem you will use group actions to establish an *extremely* important formula.

    (a) Let G be a group acting on itself by conjugation. Given any $a \in G$, prove that Stab_a is equal to
    $$C(a) = \{g \in G : ga = ag\}.$$
    This set is known as the **centralizer** of a.

    (b) Let G be a finite group acting on itself by conjugation. Since G is finite, G has finitely many orbits. Let $\mathcal{O}_{g_1}, \mathcal{O}_{g_2}, \ldots, \mathcal{O}_{g_r}$ denote the orbits that are *not* contained in $Z(G)$. Prove that
    $$|G| = |Z(G)| + \sum_{i=1}^{r} |G : C(g_i)|.$$
    This formula is known as the **class equation**.

15. *(Assignment 5)* Let p be a prime. Prove that if G is a group of order p^k for some positive integer k, then $Z(G) \neq \{e\}$.

16. *(Assignment 5)* Let p be a prime. Prove that every group of order p^2 is Abelian.

17. *(Assignment 5)* A group action $G \curvearrowright X$ is said to be **transitive** if for any two elements $x, y \in X$, there exists an $a \in G$ such that $a \cdot x = y$.

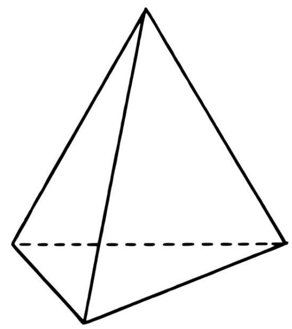
    (a) Consider the action of the group $G = GL_n(\mathbb{R})$ on the set $X = \mathbb{R}^n$ given by $A \cdot x = Ax$. Determine with justification whether or not this action is transitive.

    (b) Let G be a finite group that acts transitively on a set X. Prove that $|X|$ divides $|G|$.

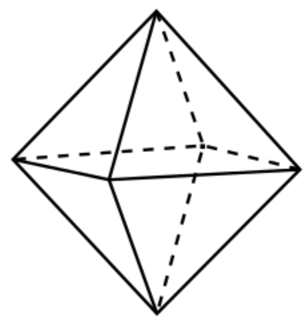
18. Prove the **Orbit-Stabilizer Theorem**: If $G \curvearrowright X$ is a group action, then for any $x \in X$, $|G : \mathrm{Stab}_x| = |\mathcal{O}_x|$. In particular, if $|G| < \infty$, then $|G| = |\mathrm{Stab}_x||\mathcal{O}_x|$.

19. Determine the number of rotational symmetries of a cube.

20. Determine the number of rotational symmetries of a tetrahedron:



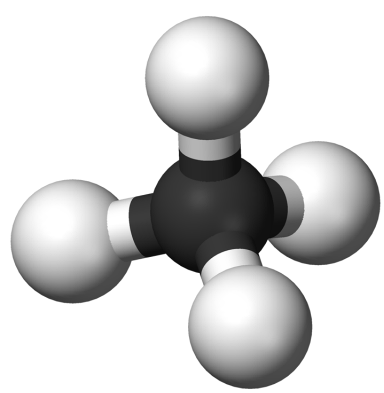
21. Determine the number of rotational symmetries of an octahedron:



22. Determine the number of rotational symmetries of a soccer ball:

A traditional soccer ball has 12 black pentagonal faces and 20 white hexagonal faces.

23. Determine the number of rotational symmetries of a methane molecule:



A methane molecule consists of a carbon atom surrounded by four equidistant hydrogen atoms.

24. Show that the only action of $\mathbb{Z}_7$ on $X = \{1, 2, 3, 4\}$ is the trivial action $g \cdot x = x$.

12 Burnside Enumeration

1. How many 2×2 chess boards can be made using black and white tiles?
   Answer: 6

2. How many 4×4 chess boards can be made using black and white tiles?
   Answer: 16456

3. Determine the number of five bead necklaces that can be made using black beads and white beads.
   Answer: 8

4. Determine the number of six bead necklaces that can be made using 1 red bead, 2 white beads, and 3 blue beads.
   Answer: 6

5. Flags are to be made using 6 vertical stripes, each coloured either red, blue, green, or yellow. How many such flags can be made? Remember that one can view the flag from the front or back.
   Answer: 2080

6. Determine the number of ways in which one can label the sides of a 6-sided die using each of the numbers 1 to 6 exactly once.
   Answer: 30

7. Any *Dungeons & Dragons* fans? Determine the number of ways in which one can label the sides of a 20-sided die using each of the numbers 1 to 20 exactly once.
   Answer: $\frac{20!}{60}$

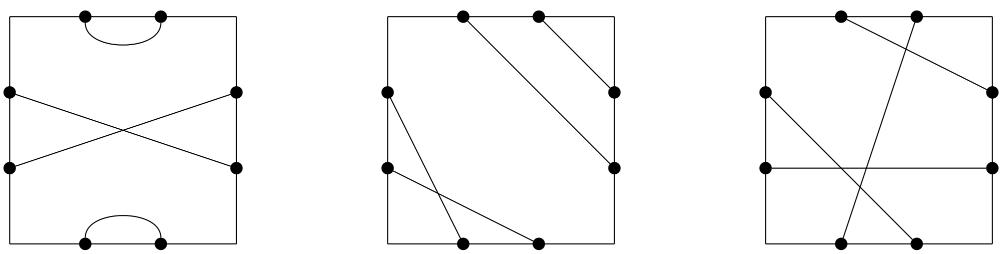
8. It's Becky's birthday! She is going to celebrate by eating a (circular) 6-piece cake. In how many ways can she place 2 red candles and 2 blue candles on the cake, assuming that multiple candles can be placed on the same piece of cake?
   Answer: 75

9. Determine the number of ways in which on can paint the faces of a tetrahedron red, blue, or green.
   Answer: 15

10. Determine the number of ways in which on can colour the edges of a tetrahedron if 3 colours are available. What if n colours are available?
    Answer: 87; $\frac{1}{12}\left(n^6 + 3n^4 + 8n^2\right)$

11. Determine the number of ways in which one can colour the vertices of a cube if exactly 8 colours are available. What if n colours are available?
Answer: 701968; $\frac{1}{24}\left(n^8 + 17n^4 + 6n^2\right)$

12. In a game of *Tsuro*, players defeat their opponents using square tiles consisting of 8 nodes (2 nodes along each edge) and 4 lines (each connecting a distinct pair of nodes). For example, here are three possible tiles from the game:



Players may rotate their tiles freely, so two tiles should be considered the same if one can be rotated into the other. Determine the number of distinct *Tsuro* tiles.
Answer: 35

13 Finite Abelian Groups

1. Let G and H be groups. Prove that the order of an element $(g, h) \in G \times H$, is $\mathrm{lcm}(|g|, |h|)$.

2. More generally, let $G_1, G_2, \ldots, G_n$ be groups, and consider an element $(g_1, g_2, \ldots, g_n)$ in $G_1 \times G_2 \times \cdots \times G_n$. Prove that $|(g_1, g_2, \ldots, g_n)| = \mathrm{lcm}(|g_1|, |g_2|, \ldots, |g_n|)$.

3. Let p be a prime. Find all Abelian groups of order p^2 up to isomorphism. Does the answer change if we remove the word "Abelian?"

4. Let p be a prime and $k \in \mathbb{N}$. Find all Abelian groups of order p^k up to isomorphism.

5. Write down all Abelian groups of order 45 up to isomorphism.

6. Write down all Abelian groups of order 36 up to isomorphism.

7. Write down all Abelian groups of order 900 up to isomorphism.

8. Let $p_1, p_2, \ldots p_k$ be distinct primes. Prove that any Abelian group of order $p_1 p_2 \cdots p_k$ is isomorphic to $\mathbb{Z}_{p_1 p_2 \cdots p_k}$.

9. Let G be an Abelian group of order 8. Suppose that $g^2 = e$ for all $g \in G$. Prove that $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

10. Let G be an Abelian group of order 75 with exactly 4 elements of order 5. Find a concrete example of a group that is isomorphic to G.

11. Let G be an Abelian group of order 100 with exactly 3 elements of order 2 and exactly 4 elements of order 5. Find a concrete example of group that is isomorphic to G.