§8 - Group Actions

§8.1 - Definitions & Examples

Up to now our study of groups has largely been about understanding the groups themselves

(e.g. How many elements of G have order k?

What are the subgroups of G?

What are the quotients of G?

Is G cyclic? Abelian? etc.)

Now we will turn our attention to studying how a group G can describe the symmetries of other sets X. Essentially, we will see how a group can permute the elements of X.

Of course, we don't want G to permute the elements of X arbitrarily, we somehow want the structure of G reflected in this permutation.

For instance, how would you like the identity $e \in G$ to permute the elements of X? If our definition is at all reasonable, it should leave everything unchanged.

It would also be reasonable to expect that for $g_1, g_2 \in G$ permuting X by the product $g_1 g_2$ should be the same as permuting by g_2 and then by g_1 (here we read right-to-left like functions.)

Definition Let G be a group and X be a set. An action of G on X is a map $\cdot : G \times X \longrightarrow X$ with the following properties:

(1) $e \cdot x = x$ for all $x \in X$.

(2) $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$ for all $g_1, g_2 \in G$ and all $x \in X$.

We write $G \curvearrowright X$ to indicate that G is acting on X.

Ex 1. The group $G = S_n$ acts on the set $X = \{1, 2, 3, \ldots, n\}$ in the natural way:

For $\sigma \in S_n$ and $i \in X$, $\sigma \cdot i = \sigma(i)$.

We'll verify that this is an action by checking properties (1) & (2):

(1) $e \cdot i = e(i) = i$ for all $i \in X$. ✓

(2) Given $\sigma, \tau \in S_n$ and $i \in X$.

$$\begin{aligned} \sigma \cdot (\tau \cdot i) &= \sigma \cdot (\tau(i)) \\ &= \sigma(\tau(i)) \\ &= (\sigma \circ \tau)(i) = (\sigma \circ \tau) \cdot i \quad ✓ \end{aligned}$$

Ex 2: The group $G = GL_n(\mathbb{R})$ acts on the set $X = \mathbb{R}^n$ by matrix-vector multiplication:

For $A \in GL_n$ and $x \in \mathbb{R}^n$, $A \cdot x = Ax$

We have that

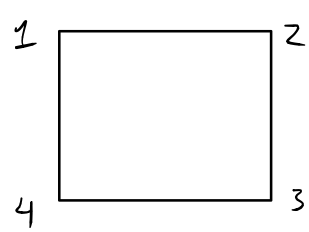
(1) $I \cdot x = Ix = x$, for all $x \in X$ ✓

(2) Given $A, B \in GL_n$ and $x \in \mathbb{R}^n$,

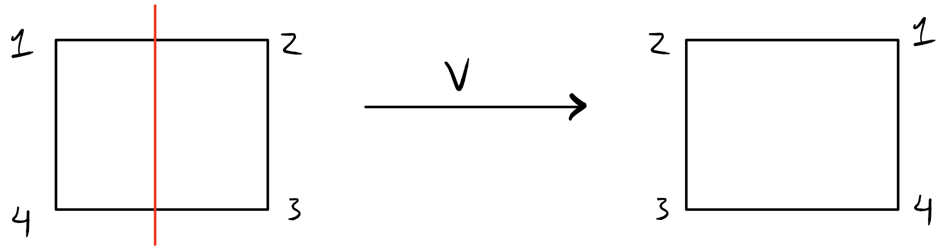
$$A \cdot (B \cdot x) = A \cdot (Bx)$$

$$= (AB)x = (AB) \cdot x$$ ✓

<u>Ex 3</u>: The group $G = D_4$ acts on the set $X = \{1, 2, 3, 4\}$, which we can think of as the set of vertices of a square

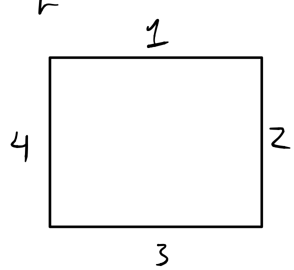


Here we have that

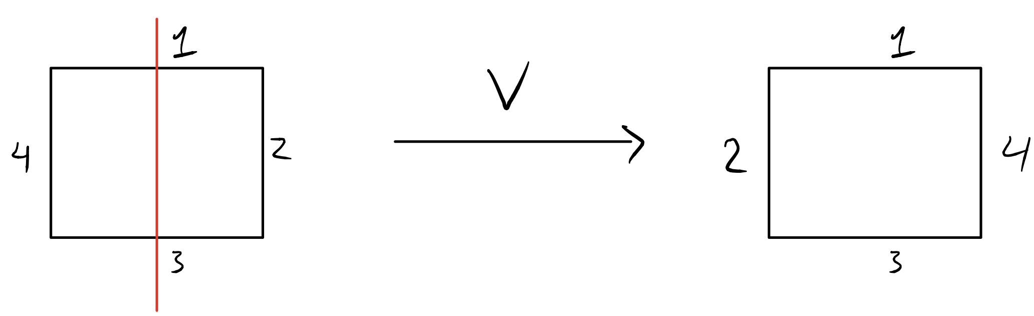


so $V \cdot 2 = 1$.

We could also think of X as the set of edges of the square



In this case,



So $V \cdot 2 = 4$ and $V \cdot 1 = 1$.

Ex 4: A group G acts on itself by left multiplication. Given $a, b \in G$, $a \cdot b = ab$

As an exercise, prove that this is an action.

Ex 5: A group G acts on itself by conjugation. Given $a, b \in G$, $a \cdot b = aba^{-1}$.

To see that this is an action, note that

(1) $e \cdot a = eae^{-1} = a$ for all $a \in G$.

(2)
$$
\begin{aligned}
a \cdot (b \cdot g) &= a \cdot (bgb^{-1}) \\
&= a(bgb^{-1})a^{-1} \\
&= (ab)g(ab)^{-1} = (ab) \cdot g.
\end{aligned}
$$

Thus, this is indeed an action.

It will sometimes be helpful to view an action $G \curvearrowright X$ in a different way. Suppose that G is a group acting on a set X. For $g \in G$ fixed, consider the map

$$\boxed{\begin{aligned} \psi_g : X &\longrightarrow X \\ x &\longmapsto g \cdot x \end{aligned}}$$

We can show that ψ_g is a bijection on X. Indeed, given $x \in X$ we have that

$$\psi_g(g^{-1} \cdot x) = g \cdot (g^{-1} \cdot x)$$

$$= (gg^{-1}) \cdot x$$

$$= e \cdot x = x$$

So ψ_g is surjective. To see that ψ_g is injective, let $x, y \in X$. Then

$$\begin{aligned}
\psi_g(x) = \psi_g(y) &\Rightarrow g \cdot x = g \cdot y \\
&\Rightarrow g^{-1} \cdot (g \cdot x) = g^{-1} \cdot (g \cdot y) \\
&\Rightarrow (g^{-1} g) \cdot x = (g^{-1} g) \cdot y \\
&\Rightarrow e \cdot x = e \cdot y \\
&\Rightarrow x = y.
\end{aligned}$$

Thus, ψ_g is injective, so ψ_g is bijective. Consequently, ψ_g belongs to the set

$$\boxed{S_X = \{ f : X \to X \mid f \text{ is bijective} \}}$$

Exercise: For any set X, the set S_X as defined above is a group under composition. Moreover, if $|X| = n < \infty$, then $S_X \cong S_n$

With this in mind, consider once again the action of G on X. Define

$$\boxed{\begin{aligned} \Phi : G &\longrightarrow S_X \\ g &\longmapsto \tau_g \end{aligned}}$$

This Φ is a map between groups. In fact, it is a group homomorphism.

Indeed, let $g_1, g_2 \in G$. We claim that

$$\Phi(g_1 g_2) = \Phi(g_1) \circ \Phi(g_2)$$

That is, we claim that

$$\psi_{g_1 g_2} = \psi_{g_1} \circ \psi_{g_2}.$$

But of course, for $x \in X$,

$$\begin{aligned}
\psi_{g_1 g_2}(x) &= (g_1 g_2) \bullet x \\
&= g_1 \bullet (g_2 \bullet x) \\
&= g_1 \bullet \psi_{g_2}(x) \\
&= \psi_{g_1}(\psi_{g_2}(x)) = (\psi_{g_1} \circ \psi_{g_2})(x).
\end{aligned}$$

Thus, we have $\Psi_{g_1 g_2} = \Psi_{g_1} \circ \Psi_{g_2}$, so $\Phi_{g_1 g_2} = \Phi_{g_1} \circ \Phi_{g_2}$. That is, Φ is a homomorphism.

Moral: $\boxed{\begin{array}{l} \text{Every action } G \curvearrowright X \text{ gives rise to a homomorphism} \\ \qquad \begin{aligned} \Phi : G &\longrightarrow S_X \\ g &\longmapsto \Psi_g \end{aligned} \\ \text{where } \begin{aligned} \Psi_g : X &\longrightarrow X \\ x &\longmapsto g \bullet x \end{aligned} \end{array}}$

Knowing this fact is enough to prove one of the coolest results in the course:

Theorem 8.1 [Cayley's Theorem]

Every group G is isomorphic to a group of permutations. In particular if $|G| = n < \infty$ then G is isomorphic to a subgroup of S_n.

Proof: Let G act on itself by left multiplication and consider the homomorphism $\Phi : G \longrightarrow S_G$ described above. Let

By the First Isomorphism Theorem,

$$G/\ker \Phi \cong \operatorname{im} \Phi \leq S_G$$

What is $\ker \Phi$? If $g \in \ker \Phi$, then $\Phi_g = \psi_g$ is the identity of S_G. Thus, for all $a \in G$, $a = \psi_g(a) = g \cdot a = ga$.

By cancellation, $g = e$, so $\ker \Phi = \{e\}$.

$\therefore$ G is isomorphic to a subgroup of S_G $\blacksquare$

§8.2 Orbits & Stabilizers

Given a group action $G \curvearrowright X$ and an element $x \in G$, there are a few natural questions one can ask

(i) Where in X can x be sent?

(ii) Which elements of G fix x?

Thus, we make the following definitions

Definition: Let $G \curvearrowright X$ be a group action. Given $x \in X$, define

(i) the orbit of x to be the set

$$\boxed{\mathcal{O}_x = \{ g \cdot x \mid g \in G \}}$$

(ii) the stabilizer of x to be the set

$$\boxed{\mathrm{Stab}_x = \{ g \in G \mid g \cdot x = x \}}.$$

Note that the orbit $\mathcal{O}_x$ is a subset of X while the stabilizer Stab_x is a subset of G. In fact:

Proposition 8.2: If $G \curvearrowright X$ is an action, then for any $x \in X$, $\mathrm{Stab}_x \leq G$

Proof: Exercise.

Is $\mathcal{O}_x$ a subgroup of X? No!

In general, X is not even a group.

Ex: The group $G = \{e^{i\theta} : \theta \in \mathbb{R}\} \leq \mathbb{C}^*$

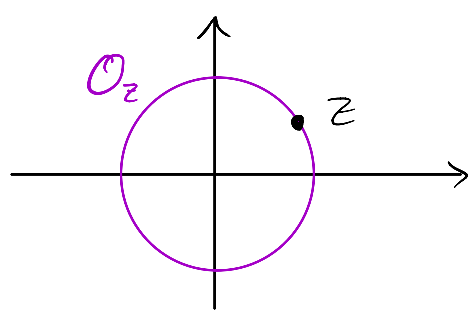
acts on the set $X = \mathbb{C}$ by multiplication:

For $z \in \mathbb{C}$, $e^{i\theta} \cdot z = e^{i\theta} z$. The orbit

of $z \in \mathbb{C}$ is

$$\mathcal{O}_z = \{e^{i\theta} z : \theta \in \mathbb{R}\}$$

$$= \{w \in \mathbb{C} \mid |w| = |z|\}.$$



Note that $\text{Stab}_0 = \{e^{i\theta} : e^{i\theta} \cdot 0 = 0\} = G$

and for $z \neq 0$, $\text{Stab}_z = \{e^{i\theta} : e^{i\theta} z = z\} = \{1\}$.

Ex: Consider the usual action of S_n on $X = \{1, 2, \dots, n\}$: $\sigma \cdot i = \sigma(i)$.

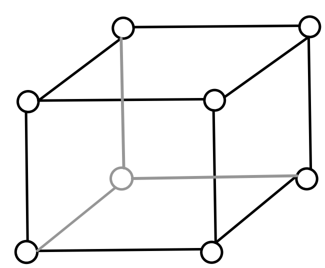
Given any $i \in X$, we can send i to any $j \in X$ using $\sigma = (i\ j)$. Thus, $\mathcal{O}_i = X$.

By definition, $\mathrm{Stab}_i = \{\sigma \in S_n \mid \sigma(i) = i\}$

$$\cong S_{n-1}.$$

Note that $|\mathcal{O}_i| = n$, $|\mathrm{Stab}_i| = (n-1)!$.

Ex: Let G be the group of all rotations of the cube



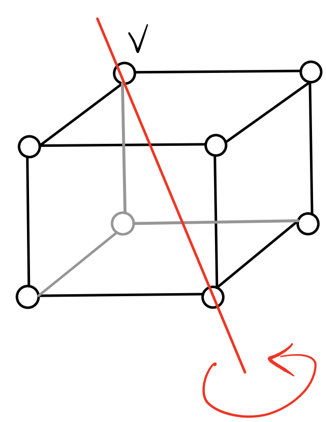
We will consider the action of G on the cube's vertices, edges, and faces.

(1) $X = \{\text{vertices of the cube}\}$

Given vertex $V \in X$, we can move V to any other vertex using a rotation.

$\therefore |O_v| = 8$

The only rotations that leave V unchanged are e and the rotations about the line through V and its opposite vertex



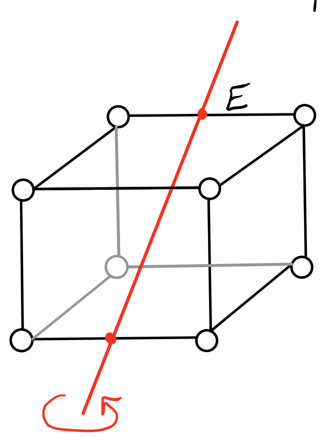
$\therefore |Stab_v| = 3$

(2) $X = \{\text{edges of the cube}\}$

Given edge $E \in X$, we can move E to any other edge using a rotation.

$\therefore |O_E| = 12$

The only rotations that leave E unchanged are e and the rotation about the line through E and its opposite edge



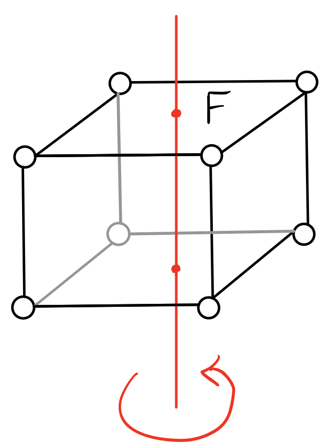
$\therefore |Stab_E| = 2$

(3) $X = \{\text{faces of the cube}\}$

Given face $F \in X$, we can move to any other face using a rotation.

$\therefore |O_F| = 6$

The only rotations that leave F unchanged are e and the rotations about the line through F and its opposite face



$\therefore |Stab_F| = 4$

Notice that in all cases $|\mathrm{Stab}_x||\mathcal{O}_x| = 24$.

It turns out that this value represents the order of the group G, and in fact, this occurs for every action $G \curvearrowright X$.

Theorem 8.3 [Orbit - Stabilizer]: Let $G \curvearrowright X$ be a group action. For every $x \in X$,

$$\boxed{|G : \mathrm{Stab}_x| = |\mathcal{O}_x|}$$

In particular, if $|G| < \infty$, then

$$\boxed{|G| = |\mathrm{Stab}_x||\mathcal{O}_x|}$$

Proof: Define $\varphi : G/\mathrm{Stab}_x \longrightarrow \mathcal{O}_x$ by

$\varphi(g \cdot \mathrm{Stab}_x) = g \cdot x$. We claim that φ

is bijective, hence $|G/\mathrm{Stab}_x| = |O_x|$.

First, we have that for all $g_1, g_2 \in G$,

$$\begin{aligned} g_1 \mathrm{Stab}_x = g_2 \mathrm{Stab}_x &\iff g_2^{-1} g_1 \in \mathrm{Stab}_x \\ &\iff (g_2^{-1} g_1) \cdot x = x \\ &\iff g_1 \cdot x = g_2 \cdot x \\ &\iff \varphi(g_1) = \varphi(g_2). \end{aligned}$$

Thus, φ is well-defined & injective.

Moreover, given $y \in O_x$ we can write $y = g \cdot x$ for some $g \in G$. Then $\varphi(g\,\mathrm{Stab}_x) = g \cdot x = y$, so φ is surjective.

$\therefore$ φ is bijective, as claimed.

The final claim for finite groups follows from Lagrange's Theorem ■

The Orbit-Stabilizer Theorem can reveal lots about a group action $G \curvearrowright X$, or about the group G itself.

Ex: What are the possible group actions of $G = \mathbb{Z}_5$ on $X = \{1,2,3\}$?

Well... given any $x \in X$, we have that

$|\mathbb{Z}_5| = |Stab_x||O_x|$, so $5 = |Stab_x||O_x|$.

Hence $|O_x| = 1$ or 5. But since $|O_x| \leq 3$, it must be that $|O_x| = 1$ for all x. That is,

$g \cdot x = x$ for all $g \in \mathbb{Z}_5$ and all $x \in X$.

$\therefore$ The only action is the trivial action.

Ex: How many rotational symmetries does a soccer ball have?

Let G be the group of all such symmetries, and let G act on the set X of all black pentagonal faces of the soccer ball.

Fix any face $F \in X$. This face can be rotated to any other such face, so $|O_F| = 12$. Moreover, the only rotations that leave F unchanged are the 5 rotations about the axis through F:

Consequently, $|\mathrm{Stab}_F| = 5$. By the Orbit-Stabilizer Theorem,

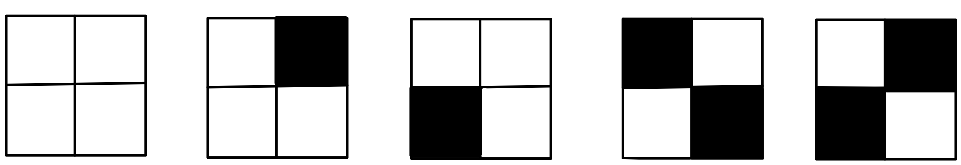
$$|G| = |\mathrm{Stab}_F||O_F|$$
$$= 5 \cdot 12$$
$$= \boxed{60}.$$

Further Applications - Burnside's Lemma

We will now see how group actions can be used to solve some neat counting problems!

Ex 1: How many different ways can we make a 2×2 chess board using black and white squares?

Here are some examples



etc...

#1 #2 #3 #4 #5

Hold on, some of these boards are really the same

For instance, #2 can be rotated to #3. So, we should consider two boards to be the same if one can be rotated into the other. That is, our problem may be restated as follows:

If G is the group of rotations of a square and X is the set of all $2^4 = 16$ 2×2 chess boards, how many orbits does the action $G \curvearrowright X$ have?

Burnside's Lemma gives us a way to count these orbits efficiently. First, we'll need the following proposition.

<u>Proposition 8.4</u>: Let $G \curvearrowright X$ be a group

action. The orbits of the action partition X. That is

(a) $X = \bigcup_{x \in X} O_x$

(b) if $x, y \in X$, then $O_x = O_y$ or $O_x \cap O_y = \emptyset$.

Proof: Assignment 5.

Lemma 8.5 [Burnside]: Let G be a finite group acting on a finite set X. If N is the number of orbits, then

$$\boxed{N = \frac{1}{|G|} \sum_{g \in G} |\mathrm{Fix}(g)|}$$

where for $g \in G$, $\underline{\mathrm{Fix}(g) = \{x \in X \mid g \cdot x = x\}}$

Proof: Let n be the number of pairs $(g, x) \in G \times X$ such that $g \cdot x = x$. First note that for a fixed $g \in G$, the number of such pairs (g, x) is $|\mathrm{Fix}(g)|$, so

$$\boxed{n = \sum_{g \in G} |\mathrm{Fix}(g)|.}$$

Also note that for fixed $x \in X$, the number of such pairs (g, x) is $|\mathrm{Stab}_x|$, so

$$\boxed{n = \sum_{x \in X} |\mathrm{Stab}_x| = \sum_{x \in X} \frac{|G|}{|O_x|} = |G| \sum_{x \in X} \frac{1}{|O_x|}}$$

But for any $y \in O_x$ we have $O_y = O_x$, so

$$\sum_{y \in O_x} \frac{1}{|O_y|} = \underbrace{\frac{1}{|O_x|} + \frac{1}{|O_x|} + \cdots + \frac{1}{|O_x|}}_{|O_x| \text{ times}} = 1$$

So $\displaystyle\sum_{x \in X} \frac{1}{|O_x|} = N$ (number of orbits)

We conclude that $\displaystyle\sum_{g \in G} |\mathrm{Fix}(g)| = n = |G| \cdot N$

So $N = \dfrac{1}{|G|} \displaystyle\sum_{g \in G} |\mathrm{Fix}(g)|$, as claimed. $\blacksquare$

We can now attempt to solve our chess board problem. We wish to determine the number of orbits of all boards under the group G of rotations of a square. By Burnside, we must find $|\mathrm{Fix}(g)|$ for all rotations $g \in G$. Note that

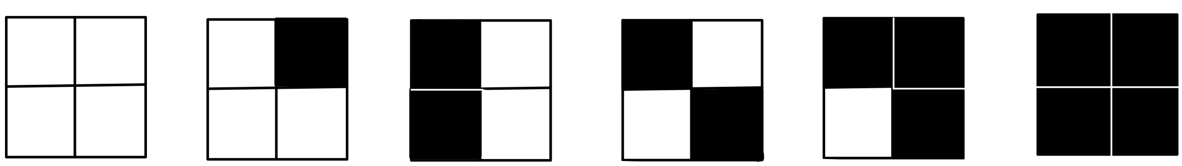
$$G = \{e, R_{90}, R_{180}, R_{270}\}.$$

g	e	R_{90}	R_{180}	R_{270}
$\lvert Fix(g)\rvert$	2^4	2	2^2	2

Thus, the number of distinct boards (orbits) is

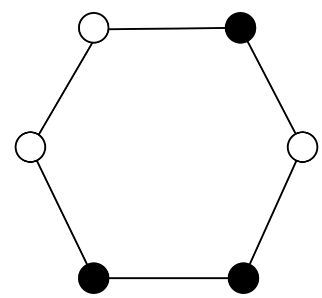
$$N = \frac{1}{|G|} \sum_{g \in G} |Fix(g)|$$

$$= \frac{1}{4}(16 + 2 + 4 + 2) = \boxed{6}$$



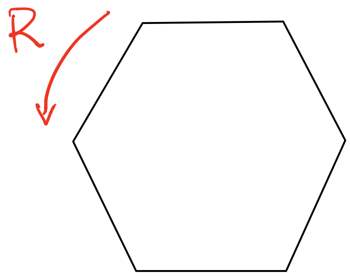
Ex: How many 6-bead necklaces can be made using 3 black beads and 3 white beads?

Solution: We can choose the location of the 3 black beads in $\binom{6}{3} = 20$ ways, and the remaining beads must be white.



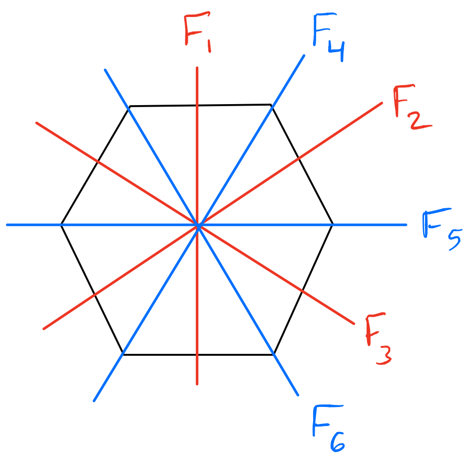
So let X be the set of these 20 possible necklaces, and let $G = D_6$ be the symmetry group of a hexagon. We consider 2 necklaces in X to be the same if they belong to the same orbit under the action $G \curvearrowright X$.

What are the symmetries in G?



Rotations:

e, R, R^2, R^3, R^4, R^5



Flips:

$F_1, F_2, F_3, F_4, F_5, F_6.$

We now compute $|Fix(g)|$ for each $g \in G$.

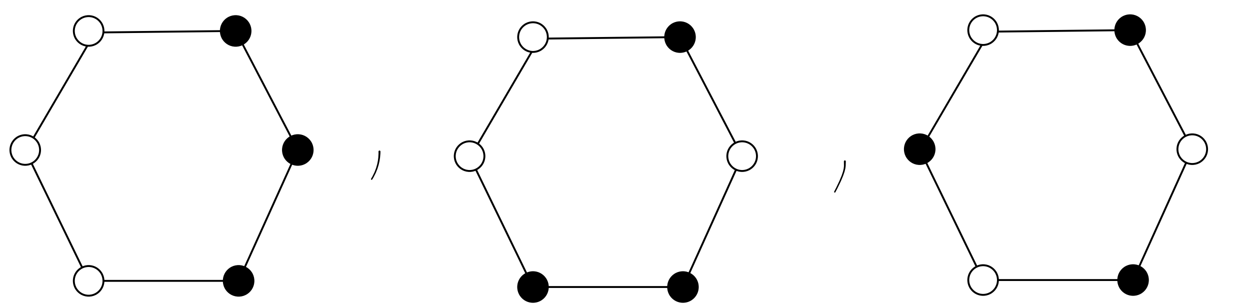
g	e	R	R^2	R^3	R^4	R^5	F_1	F_2	F_3	F_4	F_5	F_6
$Fix(g)$	20	0	2	0	2	0	0	0	0	2^2	2^2	2^2

By Burnside's Lemma, the number of orbits (i.e., the number of necklaces) is

$$\frac{1}{|G|}\sum_{g\in G}|\mathrm{Fix}(g)| = \frac{1}{12}\left(20+2+2+2^2+2^2+2^2\right)$$

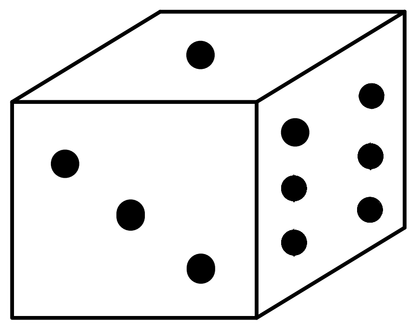
$$= \frac{1}{12}(36) = \boxed{3}$$

They are



Ex: How many ways can one label the sides of a 6-sided die using the each of the numbers 1-6 exactly once?

Solution: There are 6! ways to put the numbers on, but some of the dice may be the same after rotation



Let X be the set of all $6! = 720$ possible dice and let G be the group of rotations of the cube. We consider two dice to be the same if they are in the same orbit of the action $G \curvearrowright X$.

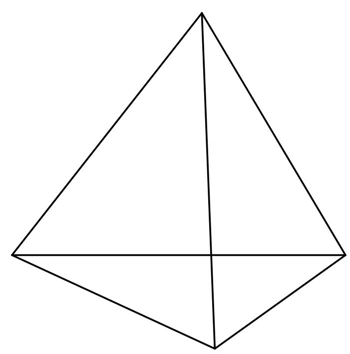
Note that the only group element that fixes every face of the cube is e. Since all faces are marked differently, we have that $|\mathrm{Fix}(g)| = 0 \ \forall g \neq e$ and $|\mathrm{Fix}(e)| = |X| = 720$.

By Burnside's Lemma, the number of orbits (i.e., the number of dice) is

$$\frac{1}{|G|} \sum_{g \in G} |\mathrm{Fix}(g)| = \frac{1}{24}(720) = \boxed{30}$$

Ex: How many ways can one paint the edges of a tetrahedron red, blue, or green?

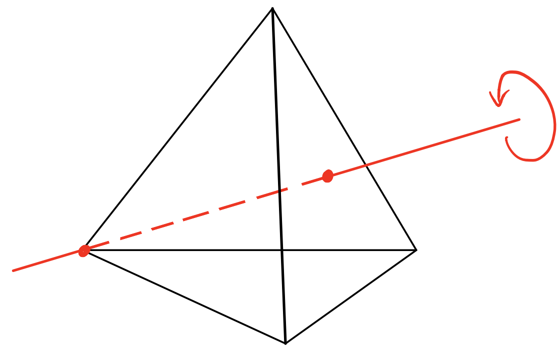
Solution: There are 3^6 different ways to paint the edges, but some of these colourings may be the same after rotation.



Let G be the group of all rotational symmetries of the tetrahedron and X be the set of all 3^6 possible colourings.

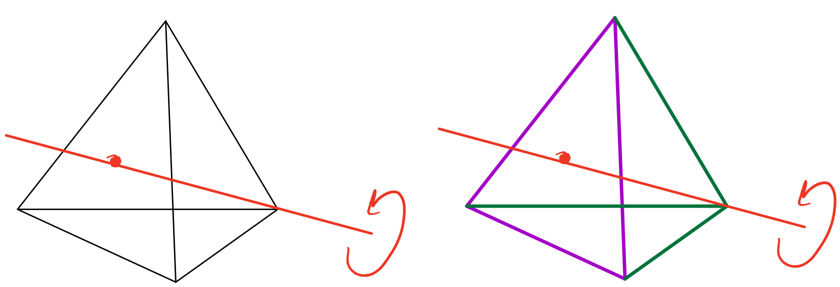
Since we must compute $|Fix(g)|$ for each $g \in G$ we should first try to understand what the rotations in G look like.

Note that G acts on the faces of the tetrahedron. For a fixed face F there are 3 rotations that fix F, so $|\mathrm{Stab}_F| = 3$. We can send F to any other face, so $|O_F| = 4$. By Orbit-Stabilizer, $|G| = 3 \cdot 4 = 12$. The 12 possible rotations are as follows:



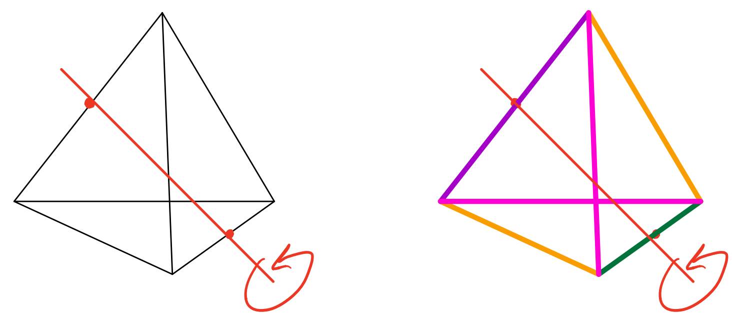
$\underline{1 \text{ identity } e}$, $|\mathrm{Fix}(e)| = 3^6$

$\underline{8 \text{ rotations } \tau \text{ about vertex and opposite face.}}$



$|\mathrm{Fix}(\tau)| = 3^2$

3 rotations σ about opposite edges.



$|\mathrm{Fix}(\sigma)| = 3^4$

We have

	1	8	3
g	e	τ	σ
$\lvert\mathrm{Fix}(g)\rvert$	3^6	3^2	3^4

So $N = \dfrac{1}{12}\left(1\cdot 3^6 + 8\cdot 3^2 + 3\cdot 3^4\right) = \boxed{87}$

The Class Equation

Let G be a finite group and let G act on itself by conjugation: $a \bullet b = aba^{-1}$

Let $\mathcal{O}_{g_1}, \mathcal{O}_{g_2}, \dots, \mathcal{O}_{g_r}$ denote the disjoint orbits of the action that are not contained in $Z(G)$. Using Proposition 8.4, one can prove that

$$\boxed{|G| = |Z(G)| + \sum_{i=1}^{r} |G : \mathrm{Stab}_{g_i}|}$$

where
$$\begin{aligned}\mathrm{Stab}_{g_i} &= \{a \in G : a g_i a^{-1} = g_i\} \\ &= \{a \in G : a g_i = g_i a\} \\ &= C(g_i) \quad (\text{Centralizer of } g_i)\end{aligned}$$

The equation

$$\boxed{|G| = |Z(G)| + \sum_{i=1}^{r} |G : C(g_i)|}$$

is called the class equation, and it has many remarkable consequences:

Corollary 8.6 Let p be a prime.

(1) If G is a group of order p^k for some $k \geq 1$, then $Z(G) \neq \{e\}$.

(2) If G is a group of order p^2, then G is Abelian.

The details are left to the assignment.

Here is another amazing application.

Theorem 8.7 [Cauchy's Theorem]

If G is a finite group and p is a prime that divides $|G|$, then G contains an element of order p.

Proof: By induction, assume that the result holds for groups of order $< |G|$.

Case I: $p \mid |Z(G)|$

By Cauchy's theorem in the Abelian case, $Z(G)$, and hence G, has an element of order p.

Case II: $p \nmid |Z(G)|$

Let G act on itself by conjugation, and let $O_{g_1}, O_{g_2}, \ldots, O_{g_r}$ be the distinct orbits not contained in $Z(G)$. By the

Class equation,

$$|Z(G)| = |G| - \sum_{i=1}^{r} |G : C(g_i)|,$$

and since $p \nmid |Z(G)|$, there must be an integer k such that $p \nmid |G : C(g_k)|$.

Since p divides $|G|$ yet p does not divide $|G : C(g_k)| = \dfrac{|G|}{|C(g_k)|}$, it must be that p divides $|C(g_k)|$. Note that $C(g_k)$ is a group and $C(g_k) \neq G$ (otherwise $ag_k = g_k a$ $\forall a \in G$, so $g_k \in Z(G)$ — contradiction). By induction, $C(g_k)$ and hence G, contains an element of order p. ■