§9 - Finite Abelian Groups

Throughout this course we have sought to understand the different types of groups that can exist by studing their various properties. In some special cases, we have been able to completely classify certain types of groups.

e.g. If G is cyclic and $|G| = n$, then $G \cong \mathbb{Z}_n$.

e.g. If G is Abelian and $|G| = pq$ (where p, q are distinct primes) then $G \cong \mathbb{Z}_{pq}$.

In general, obtaining a classification of all finite groups is an insanely challenging task. Mathematicians proved that every finite group G can be written as a direct product of what are called "simple" groups. Remarkably, a classification of all finite simple groups was obtained in the last 30 years. The classification is tens of thousands of pages long, and was written by about 100 different authors! Needless to say, this topic is well beyond the scope of our course. A much more accessible result is the classification of all finite Abelian groups.

The Fundamental Theorem of Finite Abelian Groups.

Every finite Abelian group G is isomorphic to

$$\mathbb{Z}_{p_1^{n_1}} \times \mathbb{Z}_{p_2^{n_2}} \times \cdots \times \mathbb{Z}_{p_k^{n_k}}$$

where $p_1, p_2, \ldots, p_k$ are (not necessarily distinct) primes dividing $|G|$ and $n_1, n_2, \ldots, n_k$ are positive integers. The number of terms (k) and positive integers $n_1, n_2, \ldots, n_k$ are uniquely determined by G.

Ex: If G is an Abelian group of order $8 = 2^3$, then G is isomorphic to $\mathbb{Z}_{2^3}$, $\mathbb{Z}_{2^2} \times \mathbb{Z}_2$, or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Ex: If G is an Abelian group of order $18 = 2 \cdot 3^2$, then G is isomorphic to

$$\mathbb{Z}_2 \times \mathbb{Z}_{3^2} \quad \text{or} \quad \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3.$$

Remark: What about $\mathbb{Z}_{18}$? or $\mathbb{Z}_6 \times \mathbb{Z}_3$?

Well, recall that $\mathbb{Z}_m \times \mathbb{Z}_n \cong \mathbb{Z}_{mn}$ if and only if $\gcd(m,n) = 1$, so $\mathbb{Z}_2 \times \mathbb{Z}_{3^2} \cong \mathbb{Z}_{18}$

$$\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \cong \mathbb{Z}_6 \times \mathbb{Z}_3$$

Ex: If G is an Abelian group of order $200 = 2^3 \cdot 5^2$, then G is isomorphic to

$$\mathbb{Z}_{2^3} \times \mathbb{Z}_{5^2} \quad (\cong \mathbb{Z}_{200})$$

$$\text{or} \quad \mathbb{Z}_{2^2} \times \mathbb{Z}_2 \times \mathbb{Z}_{5^2} \quad (\cong \mathbb{Z}_4 \times \mathbb{Z}_{50} \cong \mathbb{Z}_2 \times \mathbb{Z}_{100})$$

or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{5^2}$

or $\mathbb{Z}_{2^3} \times \mathbb{Z}_5 \times \mathbb{Z}_5$

or $\mathbb{Z}_{2^2} \times \mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_5$

or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_5$

Ex: Suppose that G is an Abelian group of order $36 = 2^2 \cdot 3^2$. If G has exactly three elements of order 2 and exactly two elements of order 3, find a group isomorphic to G.

Solution: It will be helpful to recall that

$$\boxed{|(a_1, a_2, \ldots, a_n)| = \operatorname{lcm}(|a_1|, |a_2|, \ldots, |a_n|)}$$

(prove this!)

Now by the Fundamental Theorem, G is isomorphic to one of

$$\mathbb{Z}_{2^2} \times \mathbb{Z}_{3^2}, \quad \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{3^2},$$

$$\mathbb{Z}_{2^2} \times \mathbb{Z}_3 \times \mathbb{Z}_3, \quad \text{or} \quad \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3.$$

Notice that

$\mathbb{Z}_{2^2} \times \mathbb{Z}_{3^2}$ contains only one element of order 2, namely $(2,0)$.

$\mathbb{Z}_{2^2} \times \mathbb{Z}_3 \times \mathbb{Z}_3$ contains more than two elements of order 3, namely $(0,1,0), (0,0,1), (0,1,1)$.

$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ contains more than two elements of order 3, namely $(0,0,1,0), (0,0,0,1), (0,0,1,1)$.

$$\therefore \boxed{G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{3^2}}$$