Math Circles - Solution Set 2
Sequences and Series cont.

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1. (a) Find the 5^{th} term in a geometric sequence with first term $a_1 = 432$ and common ratio $r = -2/3$.

(b) The 7^{th} term in a geometric sequence is 864, and the 12^{th} term is 6561. What is the 4^{th} term?

Solution. (a) The n^{th} term of a geometric sequence is given by $a_n = a_1 r^{n-1}$. Thus,

$$a_5 = a_1 r^{5-1} = 432 \cdot (-2/3)^4 = \underline{256/3}.$$

(b) The 7^{th} and 12^{th} terms of the sequence are given by the following equations:

$$\begin{cases} a_7 &= a_1 r^6 &= 864 \\ a_{12} &= a_1 r^{11} &= 6561. \end{cases}$$

If we divide the second equation by the first, we obtain

$$\frac{a_1 r^{11}}{a_1 r^6} = \frac{6561}{864} \quad \Rightarrow \quad r^5 = \frac{6561}{864}$$

$$\Rightarrow \quad r = \sqrt[5]{\frac{6561}{864}}$$

$$\Rightarrow \quad r = \frac{3}{2}.$$

Note that since $a_4 r^3 = a_7 = 864$, we can compute the fourth term as follows:

$$a_4 = \frac{864}{r^3} = \frac{864}{(3/2)^3} = \underline{256}.$$

2. Each week grandma triples the number of raisins in her cookies. In the first week, she uses only 4 raisins. In total, how many raisins are used in the first 13 weeks?

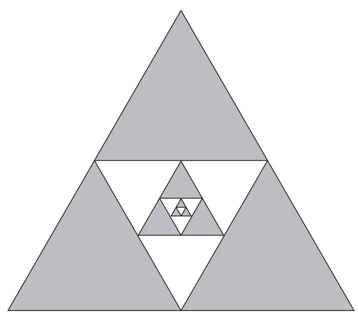
Solution. We can describe the number of raisins in grandma's cookies using the sequence

$$4\,,\ 4\cdot 3\,,\ 4\cdot 3^2\,,\ 4\cdot 3^3\,,\ \ldots$$

This is a geometric sequence with first term $a_1 = 4$ and common ratio $r = 3$. To determine the number of raisins used in the first 13 weeks, we must compute the sum of the first $n = 13$ terms in this sequence. Using the formula for the sum of a finite geometric series, we get

$$4 + 4\cdot 3 + 4\cdot 3^2 + \cdots + 4\cdot 3^{12} \;=\; \frac{4\,(1-3^{13})}{1-3} \;=\; \underline{3\,188\,644}.$$

3. The largest triangle in the figure below is equilateral with area 1. It is decomposed into infinitely many smaller equilateral triangles as shown.



Find the total area of the shaded triangles.

Solution. Let T denote the largest triangle in the figure above. Since T has area 1, it must be that the three largest shaded triangles each have area $1/4$.

What is the area of each of the *second largest* shaded triangles? Well, the part of T that is not covered by the three largest shaded triangles has area $1/4$. This region is made up of three white triangles (each with area $1/4 \cdot 1/4 = 1/16$), and a central triangular region (also with area $1/16$). Since this central region is made up of the three second largest shaded triangles and a smaller copy of the figure T, it must be that each of the second largest shaded triangles has area $1/4 \cdot 1/16 = 1/64$.

We can continue in this way to find that each of the third largest shaded triangles has area $1/256$. It becomes apparent that the areas form a geometric sequence

$$\frac{1}{4}\,,\ \frac{1}{64}\,,\ \frac{1}{265}\,,\ \ldots$$

with first term $a_1 = 1/4$ and common ratio $r = 1/16$. Since there are three triangles of each size, the total area of the shaded region is

$$\begin{aligned} 3\left(\frac{1}{4}+\frac{1}{64}+\frac{1}{256}+\cdots\right) &= 3\left(\frac{1/4}{1-1/16}\right) \\ &= 3\left(\frac{1/4}{15/16}\right) \\ &= 3\left(\frac{1/4}{15/16}\right) \\ &= \underline{4/5}. \end{aligned}$$

4. (a) Prove that $0.99999\ldots = 0.\overline{9}$ is equal to 1.

(b) Use a geometric series to write $1.1272727\ldots = 1.1\overline{27}$ as a fraction.

(c) Use a geometric series to write $0.\overline{632019}$ as a fraction.

Solution. (a) Observe that

$$\begin{aligned} 0.99999\ldots &= 0.9 + 0.09 + 0.009 + 0.0009 + \cdots \\ &= 9\left(10^{-1} + 10^{-2} + 10^{-3} + \cdots\right). \end{aligned}$$

The bracketed series is geometric with first term $a_1 = 10^{-1}$ and common ratio $r = 10^{-1}$. By our formula for the sum of an infinite geometric series, we have

$$\begin{aligned} 0.99999\ldots &= 9\left(\frac{10^{-1}}{1-10^{-1}}\right) \\ &= 9\left(\frac{1/10}{9/10}\right) \\ &= 9\left(\frac{1}{9}\right) \\ &= 1 \end{aligned}$$

(b) Observe that

$$\begin{aligned} 1.1272727\ldots &= 1.1 + (0.027 + 0.00027 + 0.0000027 + \cdots) \\ &= 1.1 + 27\left(10^{-3} + 10^{-5} + 10^{-7} + \cdots\right). \end{aligned}$$

The bracketed series is geometric with first term $a_1 = 10^{-3}$ and common ratio $r = 10^{-2}$. By our formula for the sum of an infinite geometric series, we have

$$\begin{aligned} 1.1272727\ldots &= 1.1 + 27\left(\frac{10^{-3}}{1-10^{-2}}\right) \\ &= \frac{11}{10} + 27\left(\frac{1/10^3}{(10^2-1)/10^2}\right) \\ &= \frac{11}{10} + 27\left(\frac{1}{990}\right) \\ &= \frac{11}{10} + \frac{3}{110} \\ &= \underline{62/55}. \end{aligned}$$

(c) Observe that

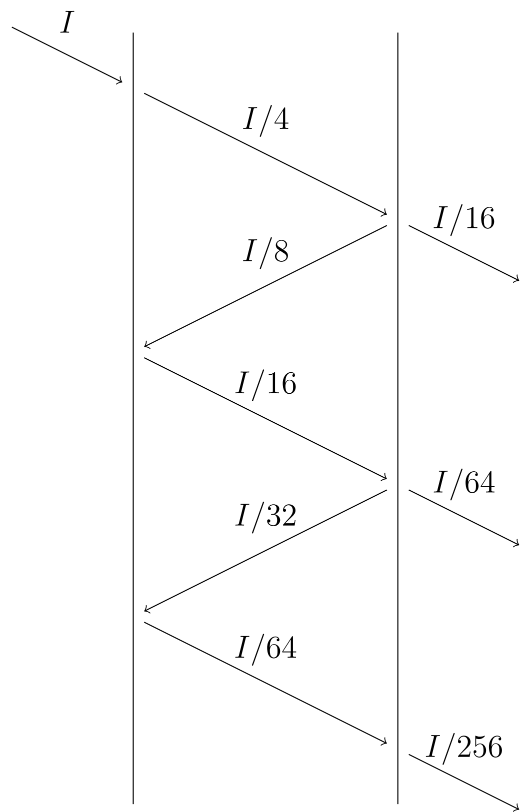
$$\begin{aligned} 0.\overline{632019} &= 0.632019 + 0.000000632019 + \cdots \\ &= 632\,019\left(10^{-6} + 10^{-12} + 10^{-18} + \cdots\right). \end{aligned}$$

The bracketed series is geometric with first term $a_1 = 10^{-6}$ and common ratio $r = 10^{-6}$. By our formula for the sum of an infinite geometric series, we have

$$\begin{aligned} 0.\overline{632019} &= 632\,019\left(\frac{10^{-6}}{1-10^{-6}}\right) \\ &= 632\,019\left(\frac{1/10^6}{(10^6-1)/10^6}\right) \\ &= 632\,019/999\,999 \\ &= \underline{210\,673/333\,333} \end{aligned}$$

5. When light hits a certain pane of glass, the glass reflects one half of the light, absorbs one fourth of the light, and transmits one fourth. A window is made of two panes of this glass separated by a small gap. If light of intensity I shines directly onto the window, what fraction is transmitted to the other side of the double pane?

Solution. This situation can be modelled by the following diagram:



If the process continues, the total amount of light passing through the second pane is

$$I\left(\frac{1}{16}+\frac{1}{64}+\frac{1}{256}+\cdots\right).$$

This is a geometric sequence with first term $a_1 = 1/16$ and common ratio $r = 1/4$. This is therefore equal to

$$I\left(\frac{1/16}{1-1/4}\right) = I\left(\frac{1/16}{3/4}\right) = \underline{I/12}$$

6. Consider the series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+2}\right)$.

(a) Find $\displaystyle\sum_{n=1}^{10}\left(\frac{1}{n}-\frac{1}{n+2}\right)$, the sum of the first 10 terms in the series.

(b) Find $\displaystyle\sum_{n=1}^{m}\left(\frac{1}{n}-\frac{1}{n+2}\right)$, the sum of the first m terms in the series.

(c) Compute the exact value of the infinite series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+2}\right)$.

Solution. (a) The sum of the first 10 terms in this series is given by

$$\left(\frac{1}{1}-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\cdots+\left(\frac{1}{9}-\frac{1}{11}\right)+\left(\frac{1}{10}-\frac{1}{12}\right).$$

Notice that starting at the third term, each positive fraction introduced to the sum immediately cancels with one of the previous term. All that remains after cancellation is

$$1+\frac{1}{2}-\frac{1}{11}-\frac{1}{12}=\underline{175/132}.$$

(b) The sum of the first m terms in this series is given by

$$\left(\frac{1}{1}-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\cdots+\left(\frac{1}{m-1}-\frac{1}{m+1}\right)+\left(\frac{1}{m}-\frac{1}{m+2}\right).$$

Notice that starting at the third term, each positive fraction introduced to the sum immediately cancels with one of the previous term. All that remains after cancellation is

$$1+\frac{1}{2}-\frac{1}{m-1}-\frac{1}{m+1}=\underline{\frac{3}{2}-\frac{1}{m-1}-\frac{1}{m+1}}.$$

(c) Note that by part (b), the sum of the first m terms of this series is given by

$$\frac{3}{2}-\frac{1}{m-1}-\frac{1}{m+1}.$$

As we add more and more terms (i.e., the value of m increases), the terms $1/(m-1)$ and $1/(m+1)$ become smaller in absolute size. When m is *huuuuuge*, these terms are essentially 0. Thus,

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+2}\right)=\frac{3}{2}-0-0=\underline{3/2}.$$