Math Circles - Lesson 2
Sequences and Series cont.

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Last week we began a discussion on **sequences**, which are ordered lists of objects. The objects in the list are called the **terms** of the sequence. The notation a_n is often used to refer to the n^{th} term of a sequence.

Example.

(a) $1, 3, 5, 7, 9, \ldots$ *Here,* $a_3 =$ ________.

(b) $2, 6, 18, 54, 162, \ldots$ *Here,* $a_4 =$ ________.

(c)  $\ldots$ *Here,* $a_6 =$ ________.

The objects in a sequence can be *anything*, but we will be interested in sequences of numbers. Specifically, we will be interested in sequences of numbers that exhibit certain nice patterns.

Last time we focused on **arithmetic sequences**, where a sequence is called **arithmetic** if the difference between consecutive terms is constant. That is, each term is obtained by adding a fixed constant d to the previous term. This d is called the **common difference**.

Example.

(a) The sequence $1, 3, 5, 7, 9, \ldots$ *is arithmetic with* $d =$ _____.

(b) The sequence $44, 32, 20, 8, -4, \ldots$ *is arithmetic with* $d =$ _____.

The simple rule that defines an arithmetic sequences makes such sequences easy to work with. In fact, we made the following neat observations in Lesson 1:

Arithmetic Sequences

Consider an arithmetic sequence $a_1, a_2, a_3, \ldots$ with common difference d.

(a) For any positive integer n, the n^{th} term in the sequence is $a_n = a_1 + (n-1)d$.

(b) The sum of the first n terms in the sequence is

$$a_1 + a_2 + \cdots + a_n = na_1 + d\left(\frac{n(n-1)}{2}\right).$$

Here is one example from last time:

(I) The number of hedgehogs on Becky's farm each day can be modelled by the sequence

$$8, 11, 14, 17, 20, 23, \ldots$$

This sequence is arithmetic with $a_1 = 8$ and common difference $d = 3$.

On day 365, Becky has

$$a_{365} = a_1 + (365-1)d = 8 + 364 \cdot 3 = 1100 \text{ hedgehogs!}$$

If she feeds each hedgehog a pancake every day, the number of pancakes needed for the first 31 days is

$$\begin{aligned} a_1 + a_2 + \cdots + a_{31} &= 31 \cdot a_1 + d\left(\frac{31(31-1)}{2}\right) \\ &= 31 \cdot 8 + 3\left(\frac{31 \cdot 30}{2}\right) \\ &= 1643. \end{aligned}$$

1 Geometric Sequences

A sequence of numbers $\{a_n\}_{n=1}^{\infty}$ is called **geometric** if ________________________________

____.

Example. *The sequence* $2, 6, 18, 54, \ldots$ *is geometric. The common ratio is* $r =$ _______.

The n^{th} Term of Geometric Sequence

If a_1 is the first term of a geometric sequence and r is the common ratio, then

$a_2 =$ __________, $a_3 =$ __________, $a_4 =$ __________, etc.

In general, its n^{th} term is $a_n =$ __________.

Recall the following example from Lesson 1:

Example. *Each week your grandma doubles the number of raisins in her cookies. In the first week, she puts in just* 1 *raisin.*

How many raisins are used in week 20*?*

Let's see if we can obtain an answer to this question!

Write down the sequence for the number of raisins grandma uses each week.

This is a geometric sequence. The common ratio is $r =$ _______.

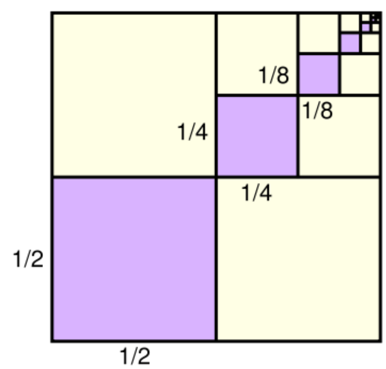
Use the formula for a_n to determine the number of raisins used on day 20.

2 Geometric Series

Recall the following example from Lesson 1:

Example. *You and a friend share a square pizza. Your share of the pizza is given by the purple squares in the diagram. Each square's side length is half that of the next largest square.*

How much pizza do you get if an infinite number of squares are cut out?



The areas of the purple squares can be described by the sequence

$$\left(\frac{1}{2}\right)^2, \left(\frac{1}{4}\right)^2, \left(\frac{1}{8}\right)^2, \left(\frac{1}{16}\right)^2, \ldots$$

or

$$\frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \frac{1}{256}, \ldots.$$

This is a geometric sequence with common ratio $r =$ _____.

Hmm... it looks like we will need to know the *sum* of the terms in a geometric sequence in order to answer this question!

Suppose first that we wish to add only finitely many terms a_1, $ra_1, r^2a_1, \ldots, r^{n-1}a_1$. Let's call the sum S_n:

$$S_n = a_1 + ra_1 + r^2a_1 + \cdots + r^{n-1}a_1 \quad .$$

Here's a neat trick! Multiply the whole sum by r:

$$\begin{aligned} S_n &= a_1 + ra_1 + r^2a_1 + \cdots + r^{n-1}a_1 \quad . \\ rS_n &= \quad\quad ra_1 + r^2a_1 + \cdots + r^{n-1}a_1 + r^na_1. \end{aligned}$$

Now subtract!

$$\begin{aligned} S_n - rS_n &= a_1 - r^na_1 \\ \Rightarrow \quad (1-r)S_n &= a_1(1-r^n) \\ \Rightarrow \quad S_n &= \frac{a_1(1-r^n)}{1-r} \end{aligned}$$

Aha! This gives us an expression for the sum of the first n terms in a geometric sequence! Such a sum is called a (finite) **geometric series.**

Finite Geometric Series

If a_1 is the first term of a geometric sequence and r is the common ratio, then the sum of the first n terms is

$$a_1 + ra_1 + r^2a_1 + \cdots + r^{n-1}a_1 = \frac{a_1(1-r^n)}{1-r}.$$

Example. *Consider the geometric sequence $\{a_n\}_{n=0}^{\infty}$ given by $a_n = 3 \cdot 2^n$. Find the sum of the first 12 terms of this sequence.*

Solution. Explicitly, this sequence is given by

$$3\,,\ 3 \cdot 2\,,\ 3 \cdot 2^2\,,\ 3 \cdot 2^3\,,\ \ldots$$

Thus, the first term is $a_1 = 3$ and the common ratio is $r = 2$. The sum of the first $n = 12$ terms is therefore

$$3 + 3 \cdot 2 + 3 \cdot 2^2 + \cdots + 3 \cdot 2^{11} = \frac{a_1(1-r^{12})}{1-r} = \frac{3(1-2^{12})}{1-2} = 12\,285.$$

We sometimes write sums like this in a more compact way:

where to stop *what to add*

$$\sum_{n=1}^{m} a_n = a_1 + a_2 + \cdots + a_m$$

where to start

So in the above example, we could have written

$$\sum_{n=1}^{12} 3 \cdot 2^{n-1} = 3 + 3 \cdot 2 + 3 \cdot 2^2 + \cdots + 3 \cdot 2^{11} = 12\,285$$

Example. *In the pizza example, what is the total area of*

(a) the first 3 *squares?*

(b) the first 10 *squares?*

Solution.

(a) The first term is $a_1 = 1/4$, and the common ratio is $r = 1/4$. Thus, we are looking for

$$\begin{aligned}
\sum_{n=1}^{3} a_1 r^{n-1} &= \sum_{n=1}^{3} \frac{1}{4} \cdot \left(\frac{1}{4}\right)^{n-1} \\
&= \frac{\frac{1}{4}\left(1 - \left(\frac{1}{4}\right)^3\right)}{1 - \frac{1}{4}} \\
&= \frac{\frac{1}{4}\left(1 - \left(\frac{1}{4}\right)^3\right)}{\frac{3}{4}} \approx 0.3281
\end{aligned}$$

(b)

Is it possible to add up *infinitely* many squares? With a little experimentation, we can see what the answer ought to be:

n	1	2	3	4	5	6	7
$a_1 + ra_1 + r^2a_1 + \cdots + r^{n-1}a_1$	0.25	0.3125	0.3281	0.332	0.333	0.3332	0.3333

Hmm... it looks like it's approaching 1/3. Is the sum of *all* the squares equal to 1/3? Let's see what our formula says!

The sum of the first n terms is

$$a_1 + ra_1 + r^2a_1 + \cdots + r^{n-1}a_1 \quad = \quad \frac{a_1\left(1 - r^n\right)}{1 - r}$$

Since $r = 1/4$, the r^n term in the numerator becomes very small for large values of n. When n is *huuuuuge*, this term is effectively 0. Thus,

$$a_1 + ra_1 + r^2a_1 + \cdots \quad = \quad \frac{a_1\left(1 - 0\right)}{1 - r} = \frac{a_1}{1 - r}\ !$$

This formula actually applies to *any* geometric sequence with $-1 < r < 1$.

Infinite Geometric Series

Suppose that a_1 is the first term of a geometric sequence and r is its common ratio. If $-1 < r < 1$, then

$$\sum_{n=1}^{\infty} a_1 r^{n-1} = a_1 + ra_1 + r^2 a_1 + \cdots = \frac{a_1}{1-r}.$$

Example. *Find the sum of the geometric series*

$$1 + \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \cdots$$

Solution. The first term is $a_1 = 1$ and the ratio is $r = 1/5$. Since $-1 < r < 1$, we can use the formula above:

$$1 + \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \cdots \quad = \quad \frac{a_1}{1-r} \quad = \quad \frac{1}{1-\frac{1}{5}} \quad = \quad \frac{5}{4}.$$

Example. *Show that the infinite sum from the pizza example is equal to* $1/3$*.*

Remark. There are *many* different types of series that one can encounter in the wild, though few behave as nicely as the arithmetic and geometric series discussed here. For example it took over 80 years for mathematicians to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

is equal to $\pi^2/6$.

There are several techniques for dealing with more complicated series, though many of these methods are beyond the scope of our discussion. Most first-year university calculus courses include a thorough treatment of this theory.