Math Circles - Problem Set 1
Introduction to Sequences and Series

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1. (a) If the sequence $7, a, b, 43, \ldots$ is arithmetic, what are the values of a and b?

(b) The 6^{th} term of an arithmetic sequence is 59, and the 21^{st} term is 14. What is the common difference?

Solution. (a) Let d be the common difference of the arithmetic sequence. Then $a = 7 + d$, $b = 7 + 2d$, and $43 = 7 + 3d$. From this last equation, we have that $3d = 43 - 7 = 36$. Thus, $d = 36/3 = 12$. Using the other two equations we can now solve for a and b:

$$a = 7 + d = 7 + 12 = \underline{19}$$

$$b = 7 + 2d = 7 + 24 = \underline{31}.$$

(b) We have that $a_6 = 59$ and $a_{21} = 14$. Using the formula $a_n = a_1 + (n-1)d$, we obtain the equations $a_6 = a_1 + 5d$ and $a_{21} = a_1 + 20d$. This means that

$$\begin{cases} a_1 + 5d = 59 \\ a_1 + 20d = 14. \end{cases}$$

Using the first equation, we see that $a_1 = 59 - 5d$. From the second equation,

$$\begin{aligned} a_1 + 20d = 14 \quad \Rightarrow \quad (59 - 5d) + 20d &= 14 \\ \Rightarrow \quad 59 + 15d &= 14 \\ \Rightarrow \quad 15d &= -45 \\ \Rightarrow \quad d &= \underline{-3}. \end{aligned}$$

2. The sum of the first n terms of a sequence is $n(n+1)(n+2)$.

(a) Write down the first 5 terms in this sequence.

(b) What is the 180^{th} term?

(c) Find an expression for the n^{th} term in the sequence.

Solution. (a) Our equation for the sum of the first n terms tells us that

$$\begin{aligned} a_1 &= 1(1+1)(1+2) &= 6 \\ a_1 + a_2 &= 2(2+1)(2+2) &= 24 \\ a_1 + a_2 + a_3 &= 3(3+1)(3+2) &= 60 \\ a_1 + a_2 + a_3 + a4 &= 4(4+1)(4+2) &= 120 \\ a_1 + a_2 + a_3 + a_4 + a_5 &= 5(5+1)(5+2) &= 210. \end{aligned}$$

Working from the top down, we have

$$\begin{array}{lclclcl} a_1 & & & & & = & \underline{6} \\ a_2 & = & 24 - a_1 & = & 24 - 6 & = & \underline{18} \\ a_3 & = & 60 - a_1 - a_2 & = & 60 - 24 & = & \underline{36} \\ a_4 & = & 120 - a_1 - a_2 - a_3 & = & 120 - 60 & = & \underline{60} \\ a_5 & = & 210 - a_1 - a_2 - a_3 - a_4 & = & 210 - 120 & = & \underline{90}. \end{array}$$

(b) We have that

$$\begin{aligned} a_1 + a_2 + a_3 + \cdots + a_{179} + a_{180} &= 180(180+1)(180+2) &= 5\,929\,560 \\ a_1 + a_2 + a_3 + \cdots + a_{179} &= 179(179+1)(179+2) &= 5\,831\,820. \end{aligned}$$

Subtract the second equation from the first to get

$$a_{180} = 5\,929\,560 - 5\,831\,820 = \underline{97\,740}.$$

(c) We can use the trick from (b) to get a general formula for a_n. In particular, since

$$\begin{aligned} a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n &= n(n+1)(n+2), \\ a_1 + a_2 + a_3 + \cdots + a_{n-1} &= (n-1)n(n+1), \end{aligned}$$

we can subtract the second equation from the first to get

$$a_n = n(n+1)(n+2) - (n-1)n(n+1) = \underline{3n(n+1)}.$$

3. (a) The sum of 100 consecutive integers is 9350. What is the largest of these integers?

(b) The sum of the first 6 terms in an arithmetic sequence is -81, and the sum of the first 40 terms is 4220. What is the 14^{th} term in the sequence?

Solution. (a) Suppose that the first integer is a_1. Since the integers are consecutive, the sequence is really

$$a_1, a_2 = a_1 + 1, a_3 = a_1 + 2, \ldots, a_{99} = a_1 + 98, a_{100} = a_1 + 99.$$

This sequence is arithmetic with common difference $d = 1$. By our formula for the sum of the terms in an arithmetic sequence, we have that

$$\begin{aligned} 9350 &= a_1 + a_2 + a_3 + \cdots + a_{100} &= 100a_1 + d\left(\frac{100(100-1)}{2}\right) \\ & &= 100a_1 + 4950. \end{aligned}$$

This means that $100a_1 = 9350 - 4950 - 4400$. Consequently, $a_1 = 44$. The largest integer in the sequence is then given by $a_{100} = a_1 + 99 = \underline{143}$.

(b) Our formula for the sum of the first n terms in an arithmetic sequence show that

$$\begin{aligned} -81 &= a_1 + a_2 + \cdots + a_6 &= 6a_1 + d\left(\frac{6(6-1)}{2}\right) &= 6a_1 + 15d \\ 4220 &= a_1 + a_2 + \cdots + a_{40} &= 40a_1 + d\left(\frac{40(40-1)}{2}\right) &= 40a_1 + 780d \end{aligned}.$$

Thus, we obtain the following system of equations:

$$\begin{cases} 6a_1 + 15d &= -81 \\ 40a_1 + 780d &= 4220 \end{cases}$$

By multiplying the first equation by $780/15 = 52$, we get

$$\begin{cases} 312a_1 + 780d &= -4212 \\ 40a_1 + 780d &= 4220 \end{cases}$$

By subtracting the second equation from the first, we see that $272a_1 = -8432$, so $a_1 = -8432/272 = -31$. Using the first equation, we have

$$15d = -81 - 6a_1 = 15d = -81 - 6(-31) = 105,$$

so $d = 105/15 = 7$.

We may now use the formula for the n^{th} term of an arithmetic sequence to determine the 14^{th} term:

$$a_{14} = a_1 + (14-1)d = -31 + 13(7) = \underline{60}.$$

4. (a) Find the sum of the first 1000 positive integers.

(b) Find the sum of the numbers between 1 and 1000 (including 1 and 1000) that are not multiples of 3.

(c) Determine the value of $1 - 2 + 3 - 4 + \cdots + 99 - 100$.

Solution. (a) We would like to find the sum of the terms in the sequence

$$1, 2, 3, \ldots, 1000.$$

Of course, this is an arithmetic sequence with $a_1 = 1$ and $d = 1$. Thus, we may use our formula for the sum of such a sequence:

$$\begin{aligned} 1 + 2 + \cdots + 1000 &= 1000(1) + 1\left(\frac{1000(1000-1)}{2}\right) \\ &= 1000 + 499\,500 \\ &= \underline{500\,500}. \end{aligned}$$

(b) Let's find the sum of the integers between 1 and 1000 that *are* multiples of 3, and then subtract this from our answer in (a). That is, we wish to find the sum of the terms in the sequence

$$3, 6, 9, 12, \ldots, 999$$

This is an arithmetic sequence whose first term is $a_1 = 3$ and common difference is $d = 3$. How many terms are there in this sequence? Since

$$999 = a_n = a_1 + (n-1)d = 3 + 3(n-1) = 3n,$$

we have $n = 333$. That is, 999 is the 333^{th} term in the sequence. Using our formula for the sum of the terms in an arithmetic sequence, we get that

$$\begin{aligned} 3 + 6 + 9 + \cdots + 999 &= 333(3) + 3\left(\frac{333(333-1)}{2}\right) \\ &= 999 + 165\,834 \\ &= 166\,833. \end{aligned}$$

We conclude that the sum of the integers between 1 and 1000 that are *not* multiples of 3 is therefore

$$500\,500 - 166\,833 = \underline{333\,667}.$$

(c) Note that

$$1 - 2 + 3 - 4 + \cdots + 99 - 100 = (1 + 3 + 5 + \cdots + 99) - (2 + 4 + 6 + \cdots + 100).$$

Thus, we can find the sum of each group separately, and then compute their difference.

We have that $1, 3, 5, \ldots, 99$ is an arithmetic sequence with $a_1 = 1$ and $d = 2$. There are 50 terms in this sequence. Using our summation formula, we get

$$\begin{aligned} 1 + 3 + 5 + \cdots + 99 &= 50a_1 + d\left(\frac{50(50-1)}{2}\right) \\ &= 50 + 2450 \\ &= 2500 \end{aligned}$$

Now consider the second sum. We have that $2, 4, 6, \ldots, 100$ is an arithmetic sequence with $a_1 = 2$ and $d = 2$. There are 50 terms in this sequence. Using our summation formula, we get

$$\begin{aligned} 2 + 4 + 6 + \cdots + 100 &= 50a_1 + d\left(\frac{50(50-1)}{2}\right) \\ &= 100 + 2450 \\ &= 2550 \end{aligned}$$

Thus,

$$\begin{aligned} 1 - 2 + 3 - 4 + \cdots + 99 - 100 &= (1 + 3 + 5 + \cdots + 99) - (2 + 4 + 6 + \cdots + 100) \\ &= 2500 - 2550 \\ &= \underline{-50}. \end{aligned}$$

5. The numbers $2, 5, 8, 11, 14, \ldots$ are written in order in a book, beginning on page 1. There are 100 numbers on each page. On what page can the number 11 111 be found?

Solution. We must determine where 11 111 occurs in this sequence. First note that the sequence is arithmetic with first term $a_1 = 2$ and common difference $d = 3$. Thus, the terms are given by $a_n = a_1 + (n-1)d$.

To determine the position of 11 111 in the sequence, we solve for n in the equation in the equation $a_n = 11\,111$. We have

$$\begin{aligned} a_n = 11\,111 \quad \Rightarrow \quad a_1 + (n-1)d &= 11\,111 \\ \Rightarrow \quad 2 + 3(n-1) &= 11\,111 \\ \Rightarrow \quad 3n &= 11\,112 \\ \Rightarrow \quad n &= 3704. \end{aligned}$$

Thus, 11 111 is the 3704^{th} term in the sequence. Since each page contains 100 numbers, we have that page 1 contains $a_1, a_2, \ldots, a_{100}$; page 2 contains $a_{101}, a_{102}, \ldots, a_{200}$; page 3 contains $a_{201}, a_{202}, \ldots, a_{300}$; etc. It follows that page 38 contains $a_{3701}, a_{3702}, \ldots, a_{3800}$, and hence contains $a_{3704} = 11\,111$.

Therefore, 11 111 is found on page $\underline{38}$.

6. (a) The 3^{rd} term in a geometric sequence is 8 and the 6^{th} term is 17 576. What is the common ratio?

(b) The 10^{th} term of a geometric sequence is -6655 and the 13^{th} term is 5. What is the common ratio?

Solution. (a) The n^{th} term of a geometric sequence is given by $a_n = a_1 r^{n-1}$. Thus, we have that

$$\begin{cases} 8 &= a_3 &= a_1 r^2 \\ 17\,576 &= a_6 &= a_1 r^5 \end{cases}$$

By multiplying the equation by r^3, we have that

$$\begin{cases} 8r^3 &= a_1 r^5 \\ 17\,576 &= a_1 r^5 \end{cases}$$

It follows $8r^3 = 17\,576$. Consequently, $r^3 = 17\,576/8 = 2197$, so

$$r = \sqrt[3]{2197} = \underline{13}.$$

(b) The n^{th} term of a geometric sequence is given by $a_n = a_1 r^{n-1}$. Thus, we have that

$$\begin{cases} -6655 &= a_{10} &= a_1 r^9 \\ 5 &= a_{13} &= a_1 r^{12} \end{cases}$$

By multiplying the equation by r^3, we have that

$$\begin{cases} -6655r^3 &= a_1 r^{12} \\ 5 &= a_1 r^{12} \end{cases}$$

It follows $-6655r^3 = 5$. Consequently, $r^3 = -5/6655 = -1/1331$, so

$$r = \sqrt[3]{\frac{-1}{1331}} = \underline{\frac{-1}{11}}.$$

7. (a) Consider the recursive sequence defined by $a_1 = 9$ and $a_n = a_{n-1} - 4$ for all $n \geq 2$. Find a formula for a_n that depends only on n.

(b) Consider the recursive sequence defined by $a_1 = 1$, $a_2 = -1$, and $a_n = \left(\frac{n-3}{n-1}\right) a_{n-2}$ for $n \geq 3$. Determine the values of a_{2019} and a_{2020}.

Solution. (a) Let's start by writing out some of the terms of this sequence:

$$\begin{array}{lcccccc} a_1 & & & & & = & 9 \\ a_2 & = & a_1 - 4 & = & 9-4 & = & 5 \\ a_3 & = & a_2 - 4 & = & 5-4 & = & 1 \\ a_4 & = & a_3 - 4 & = & 1-4 & = & -3 \\ a_5 & = & a_4 - 4 & = & -3-4 & = & -7 \end{array}$$

We can see that this sequence is arithmetic with first term $a_1 = 9$ and common difference $d = -4$. Thus, the formula for the n^{th} term is

$$\underline{a_n = a_1 + (n-1)d = 9 - 4(n-1)}.$$

(b) Let's start by writing out some of the terms of this sequence:

$$\begin{array}{lcccccc} a_1 & & & & & = & 1 \\ a_2 & & & & & = & -1 \\ a_3 & = & \left(\frac{3-3}{3-1}\right) a_1 & = & 0 \cdot 1 & = & 0 \\ a_4 & = & \left(\frac{4-3}{4-1}\right) a_2 & = & \frac{1}{3} \cdot -1 & = & \frac{-1}{3} \\ a_5 & = & \left(\frac{5-3}{5-1}\right) a_3 & = & \frac{2}{4} \cdot 0 & = & 0 \\ a_6 & = & \left(\frac{6-3}{6-1}\right) a_4 & = & \frac{3}{5} \cdot \frac{-1}{3} & = & \frac{-1}{5} \\ a_7 & = & \left(\frac{7-3}{7-1}\right) a_5 & = & \frac{4}{6} \cdot 0 & = & 0 \\ a_8 & = & \left(\frac{8-3}{8-1}\right) a_6 & = & \frac{5}{7} \cdot \frac{-1}{5} & = & \frac{-1}{7}. \end{array}$$

Notice that $a_n = 0$ whenever n is odd and $n \geq 3$. Moreover, observe that $a_{2n} = \frac{-1}{2n-1}$ for all $n \geq 1$. This means that

$$a_{2019} = \underline{0}$$

and

$$a_{2020} = a_{2(1010)} = \frac{-1}{2(1010) - 1} = \underline{\frac{-1}{2019}}.$$

Challenge Problems

8. Can a sequence with infinitely many terms be both arithmetic and geometric? If so, describe all sequences with this property.

Solution. There are indeed sequences that are both arithmetic and geometric. For example, consider the sequence

$$0, 0, 0, 0, \ldots$$

This is arithmetic with difference $d = 0$, and geometric with ratio $r = 1$ (really, you can pick r to be whatever you like!)

More generally, one could fix a constant c and consider the sequence

$$c, c, c, c \ldots$$

This is arithmetic with difference $d = 0$, and geometric with ratio $r = 1$.

Are there any *other* sequences with this property? Suppose there were a non-constant sequence that were both arithmetic and geometric. This sequence cannot be constantly 0, so there must be some non-zero term a_n.

Since such a sequence must be arithmetic (say with difference d) and geometric (say with ratio r), we can write the terms starting at a_n in two different ways:

$$a_n, a_n + d, a_n + 2d, \ldots$$

or

$$a_n, a_n r, a_n r^2, \ldots$$

It must therefore be the case that our two expressions for the second term of this sequence are the same. That is, $a_1 + d = a_1 r$. This implies that

$$d = a_n r - a_n = a_n(r - 1).$$

Now let's examine the third term of the sequence. We have that

$$a_n + 2d = a_1 r^2.$$

But since $d = a_n(r-1)$, this becomes

$$a_n + 2(a_n(r-1)) = a_n r^2$$

We can rearrange this equation and factor to obtain

$$a_n (r-1)^2 = 0.$$

Since $a_n \neq 0$, we conclude that $r - 1 = 0$. That is, $r = 1$. But since r is the ratio of our geometric sequence, it follows that each term in the sequence can be obtained by multiplying the previous term by 1. This means the sequence we started with was actually constant. Uh oh... this is a contradiction.

We conclude that the only sequences that are both arithmetic and geometric are the sequences of the form

$$c, c, c, c, \ldots$$

for some constant c.

9. Determine an expression for the sum of the first n terms in a geometric sequence. (We'll use this next lesson to solve the pizza problem!)

Solution. This problem has a cute trick!

Suppose that our geometric sequence has r as the common ratio, and let S_n denote the sum of the first n terms of the sequence. That is,

$$S_n = a_1 + a_1 r + a_1 r^2 + \cdots + a_1 r^{n-2} + a_1 r^{n-1}.$$

By multiplying both sides by r, we get

$$rS_n = a_1 r + a_1 r^2 + \cdots + a_1 r^{n-2} + a_1 r^{n-1} + a_1 r^n.$$

We can now subtract the first equation from the second to get

$$S_n - rS_n = a_1 - a_1 r^n.$$

This means that $(1-r)S_n = a_1(1-r^n)$, or

$$\underline{S_n = \frac{a_1(1-r^n)}{1-r}}.$$