Math Circles - Solution Set 2
Linear Diophantine Equations cont.

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1. Find the full list of solutions to the following LDEs:

(a) $94x + 44y = 12$

(b) $1\,002x + 954y = 42$

Solution. (a) The Euclidean algorithm proceeds as follows:

$$94 = 44(2) + 6 \quad \Rightarrow \quad \gcd(94, 44) = \gcd(44, 6) \qquad (1)$$

$$44 = 6(7) + 2 \quad \Rightarrow \quad \gcd(44, 6) = \gcd(6, 2) \qquad (2)$$

$$6 = 2(3) + 0 \quad \Rightarrow \quad \gcd(6, 2) = \gcd(2, 0) = \underline{2}.$$

The process ends and we conclude that $\gcd(94, 44) = 2$.

Since $d = \gcd(94, 44) = 2$ divides 12, a solution exists. To find it, we'll work backwards through the steps of the Euclidean algorithm:

$$\begin{aligned} 2 &= 44 - \underline{6}(7) && \text{by (2)} \\ &= 44 - [94 - 44(2)](7) && \text{by (1)} \\ &= 44(15) + 94(-7). \end{aligned}$$

We arrive at the equation

$$94(-7) + 44(15) = 2.$$

To finish, we multiply both sides of this equation by $12/2 = 6$ and obtain the following solution to our LDE:

$$94(-42) + 44(90) = 12.$$

Thus, we have that

- $a = 94$, $b = 44$,
- $d = \gcd(a, b) = 2$, and
- $(x_0, y_0) = (-42, 90)$ is a particular solution.

This means that the full list of solutions is given by

$$\begin{aligned} x = x_0 + n\left(\frac{b}{d}\right) &= -42 + n\left(\frac{44}{2}\right) \\ &= \underline{-42 + 22n} \end{aligned}$$

$$\begin{aligned} y = y_0 - n\left(\frac{a}{d}\right) &= 90 - n\left(\frac{94}{2}\right) \\ &= \underline{90 - 47n} \end{aligned}$$

where n is any integer.

(b) The Euclidean algorithm proceeds as follows:

$$\begin{aligned} 1\,002 &= 954(1) + 48 &\Rightarrow\quad & \gcd(1\,002, 954) = \gcd(954, 48) && (1) \\ 954 &= 48(19) + 42 &\Rightarrow\quad & \gcd(954, 48) = \gcd(48, 42) && (2) \\ 48 &= 42(1) + 6 &\Rightarrow\quad & \gcd(48, 42) = \gcd(42, 6) && (3) \\ 42 &= 6(7) + 0 &\Rightarrow\quad & \gcd(42, 6) = \gcd(6, 0) = \underline{6}. \end{aligned}$$

The process ends and we conclude that $\gcd(1\,002, 954) = 6$.

Since $d = \gcd(1\,002, 954) = 6$ divides 42, a solution exists. To find it, we'll work backwards through the steps of the Euclidean algorithm:

$$
\begin{aligned}
6 &= 48 - \underline{42}(1) && \text{by (3)} \\
&= 48 - [954 - 48(19)](1) && \text{by (2)} \\
&= \underline{48}(20) - 954. \\
&= [1\,002 - 954(1)](20) - 954 && \text{by (1)} \\
&= 1\,002(20) + 954(-21).
\end{aligned}
$$

We arrive at the equation

$$1\,002(20) + 954(-21) = 6.$$

To finish, we multiply both sides of this equation by $42/6 = 7$ and obtain the following solution to our LDE:

$$1\,002(140) + 954(-147) = 42.$$

Thus, we have that

- $a = 1\,002$, $b = 954$,
- $d = \gcd(a, b) = 6$, and
- $(x_0, y_0) = (140, -147)$ is a particular solution.

This means that the full list of solutions is given by

$$
\begin{aligned}
x = x_0 + n\left(\frac{b}{d}\right) &= 140 + n\left(\frac{954}{6}\right) \\
&= \underline{140 + 159n}
\end{aligned}
$$

$$
\begin{aligned}
y = y_0 - n\left(\frac{a}{d}\right) &= -147 - n\left(\frac{1\,002}{6}\right) \\
&= \underline{-147 - 167n}
\end{aligned}
$$

where n is any integer.

2. Becky wants exactly 600 of her daily calories to come from green eggs and ham. Each slice of ham has 102 calories, and each egg has 18 calories. Find all combinations of green eggs and ham that will total 600 calories.

Solution. Let x represent the number of slices of ham that Becky eats, and let y represent the number of eggs that she eats. We are looking for all *non-negative* solutions to the LDE

$$102x + 18y = 600.$$

The Euclidean algorithm proceeds as follows:

$$102 = 18(5) + 12 \quad \Rightarrow \quad \gcd(102, 18) = \gcd(18, 12) \qquad (1)$$

$$18 = 12(1) + 6 \quad \Rightarrow \quad \gcd(18, 12) = \gcd(12, 6) \qquad (2)$$

$$12 = 6(2) + 0 \quad \Rightarrow \quad \gcd(12, 6) = \gcd(6, 0) = \underline{6}.$$

The process ends and we conclude that $\gcd(102, 18) = 6$.

Since $d = \gcd(102, 18) = 6$ divides 600, a solution exists. To find it, we'll work backwards through the steps of the Euclidean algorithm:

$$\begin{aligned} 6 &= 18 - \underline{12}(1) && \text{by (2)} \\ &= 18 - [102 - 18(5)](1) && \text{by (1)} \\ &= 18(6) + 102(-1). \end{aligned}$$

We arrive at the equation

$$102(-1) + 18(6) = 6.$$

To finish, we multiply both sides of this equation by $600/6 = 100$ and obtain the following solution to our LDE:

$$102(-100) + 18(600) = 600.$$

Thus, we have that

- $a = 102$, $b = 18$,
- $d = \gcd(a, b) = 6$, and
- $(x_0, y_0) = (-100, 600)$ is a particular solution.

This means that the full list of solutions is given by

$$\begin{aligned} x = x_0 + n\left(\frac{b}{d}\right) &= -100 + n\left(\frac{18}{6}\right) \\ &= \underline{-100 + 3n} \end{aligned}$$

$$\begin{aligned} y = y_0 - n\left(\frac{a}{d}\right) &= 600 - n\left(\frac{102}{6}\right) \\ &= \underline{600 - 17n} \end{aligned}$$

where n is any integer.

Let's now restrict our attention to the non-negative solutions:

$$\begin{aligned} x \geq 0 \quad &\Rightarrow \quad 3n \geq 100 \\ &\Rightarrow \quad n \geq \frac{100}{3} \approx 33.33 \\ &\Rightarrow \quad n \geq 34 \qquad \text{(since } n \text{ is an integer).} \end{aligned}$$

$$\begin{aligned} y \geq 0 \quad &\Rightarrow \quad 17n \leq 600 \\ &\Rightarrow \quad n \leq \frac{600}{17} \approx 35.29 \\ &\Rightarrow \quad n \leq 35 \qquad \text{(since } n \text{ is an integer).} \end{aligned}$$

Below are the 2 valid solutions to this equation:

n	34	35
$x = -100 + 3n$	2	5
$y = 600 - 17n$	22	5

Conclusion: Becky can eat 2 slices of ham and 22 eggs, or 5 slices of ham and 5 eggs.

3. A strange virus that turns people into goombas has infected the entire city of Toronto. Luckily, one of the city's top scientists was able to send you a distress email before she was infected. The email contains the following information on creating a cure:

- Each person can be cured by consuming exactly 1 812 mg of chemical Z.
- Chemical Z can only be found in pickles and marshmallows.

- Each pickle contains 312 mg of chemical Z.
- Each marshmallow contains 252 mg of chemical Z.

How many pickles and marshmallows should each person be given in order to restore the city to normal?

Solution. Let x represent the number pickles consumed, and let y represent the number of marshmallows consumed. We are looking for a *non-negative* solution to the LDE

$$312x + 252y = 1\,812.$$

The Euclidean algorithm proceeds as follows:

$$312 = 252(1) + 60 \quad \Rightarrow \quad \gcd(312, 252) = \gcd(252, 60) \qquad (1)$$

$$252 = 60(4) + 12 \quad \Rightarrow \quad \gcd(252, 60) = \gcd(60, 12) \qquad (2)$$

$$60 = 12(5) + 0 \quad \Rightarrow \quad \gcd(60, 12) = \gcd(12, 0) = \underline{12}.$$

The process ends and we conclude that $\gcd(312, 252) = 12$.

Since $d = \gcd(312, 252) = 12$ divides $1\,812$, a solution exists. To find it, we'll work backwards through the steps of the Euclidean algorithm:

$$\begin{aligned} 12 &= 252 - \underline{60}(4) && \text{by (2)} \\ &= 252 - [312 - 252(1)](4) && \text{by (1)} \\ &= 252(5) + 312(-4). \end{aligned}$$

We arrive at the equation

$$312(-4) + 252(5) = 12.$$

To finish, we multiply both sides of this equation by $1\,812/12 = 151$ and obtain the following solution to our LDE:

$$312(-604) + 252(755) = 1\,812.$$

Thus, we have that

- $a = 312$, $b = 252$,
- $d = \gcd(a, b) = 12$, and
- $(x_0, y_0) = (-604, 755)$ is a particular solution.

This means that the full list of solutions is given by

$$\begin{aligned} x = x_0 + n\left(\frac{b}{d}\right) &= -604 + n\left(\frac{252}{12}\right) \\ &= \underline{-604 + 21n} \end{aligned}$$

$$\begin{aligned} y = y_0 - n\left(\frac{a}{d}\right) &= 755 - n\left(\frac{312}{12}\right) \\ &= \underline{755 - 26n} \end{aligned}$$

where n is any integer.

Let's now restrict our attention to the non-negative solutions:

$$\begin{aligned} x \geq 0 \quad &\Rightarrow \quad 21n \geq 604 \\ &\Rightarrow \quad n \geq \frac{604}{21} \approx 28.76 \\ &\Rightarrow \quad n \geq 29 \qquad \text{(since } n \text{ is an integer).} \end{aligned}$$

$$\begin{aligned} y \geq 0 \quad &\Rightarrow \quad 26n \leq -755 \\ &\Rightarrow \quad n \leq \frac{755}{26} \approx 29.04 \\ &\Rightarrow \quad n \leq 29 \qquad \text{(since } n \text{ is an integer).} \end{aligned}$$

Neat! We've shown that the only valid solution occurs when $n = 29$. We have that

$$x = -604 + 21n = -604 + 21(29) = 5$$

and

$$y = 755 - 26n = 755 - 26(29) = 1.$$

Conclusion: Each person should consume 5 pickles and 1 marshmallow.

Challenge Problems

4. Let a, b, and c be positive integers. Show that the number of non-negative integer solutions to the LDE

$$ax + by = c$$

cannot exceed c/a or c/b.

Solution. The easiest way to see this is by thinking of the solutions to this LDE as integer points on a line (see notes). By rearranging things a bit, the equation can be written as

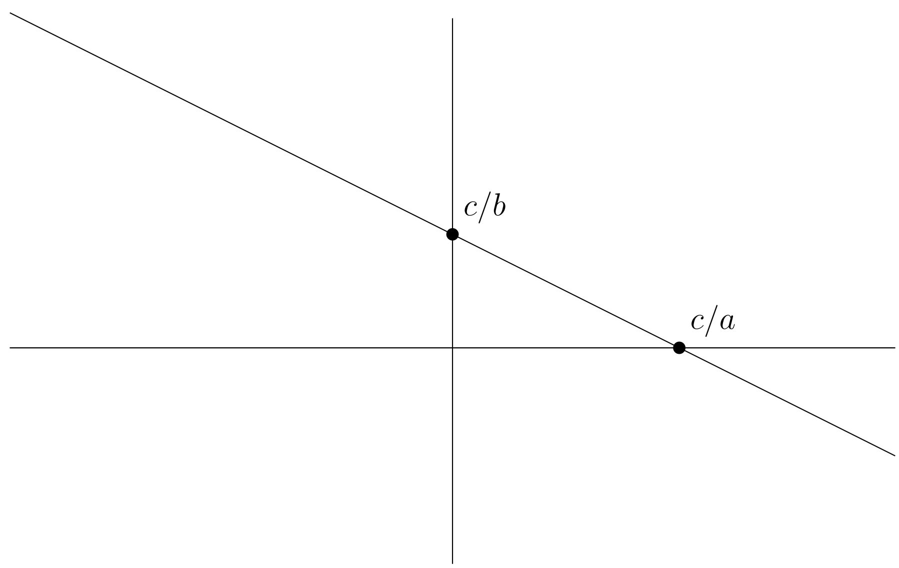
$$ax + by = c \quad \longrightarrow \quad by = -ax + c \quad \longrightarrow \quad \boxed{y = \frac{-a}{b}x + \frac{c}{b}}.$$

This is the equation of a line with slope $-a/b$ and y-intercept c/b.

What is the x-intercept of this line? We can find it by setting $y = 0$ and solving for x:

$$0 = \frac{-a}{b}x + \frac{c}{b} \quad \Rightarrow \quad \frac{a}{b}x = \frac{c}{b} \quad \Rightarrow \quad x = \frac{bc}{ab} = \frac{c}{a}.$$

So this line has its x-intercept at $x = c/a$. The picture we should have in mind is something like this:



Notice that the non-negative solutions are exactly the integer points that lie on the line segment from $\left(\frac{c}{a}, 0\right)$ to $\left(0, \frac{c}{b}\right)$. How many of these points can there be? Since

$$0 \leq x \leq \frac{c}{a},$$

there are at most c/a such points. Likewise, since

$$0 \leq y \leq \frac{c}{b},$$

there can be at most c/b such points. So the number of non-negative solutions cannot exceed c/a or c/b.