Math Circles - Lesson 2
Linear Diophantine Equations cont.

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March 7, 2018

Last week we discussed **linear Diophantine equations** (**LDEs**), which are equations of the form

$$ax + by = c$$

for some integers a, b, and c. The important thing about Diophantine equations is that their solutions have to be integers!

Here are some examples from last time:

(I) $8x + 3y = 10$ has a solution given by $8(-10) + 3(30) = 10$.

(II) $14x + 35y = 4$ has *no solutions*. Why? Because if we divide both sides by 7, we get

$$2x + 5y = \frac{4}{7}.$$

Since the left-hand side is an integer and the right-hand side is not, no solutions can exist.

Example (II) shows that the existence of solutions to $ax + by = c$ depends on the common divisors of a and b. In particular, it was important for us to be able to find $\gcd(a, b)$, the **greatest common divisor** of a and b. To do this, we used the following tools:

The Division Algorithm. Let a and b be integers with $b > 0$. There are unique integers q (the *quotient*) and r (the *remainder*) such that

$$a = bq + r \quad \text{and} \quad 0 \leq r < b.$$

Furthermore, $\gcd(a, b) = \gcd(b, r)$.

The Euclidean Algorithm.

Step 1: Arrange a and b so that $a \geq b$.

Step 2: Write $a = bq + r$ where $0 \leq r < b$.

Step 3: If $r = 0$, then stop! We get $\gcd(a, b) = \gcd(b, 0) = b$.

Step 4: Replace (a, b) with (b, r) and return to Step 1.

These algorithms led to the following result on solutions to $ax + by = c$:

Theorem. The LDE

$$ax + by = c$$

has a solution if and only if $\gcd(a, b)$ divides c.

Using the Euclidean algorithm and working backwards, we get an equation of the form

$$ax_0 + by_0 = \gcd(a, b).$$

The solution to the LDE can then be obtained by multiplying both sides of this equation by $\frac{c}{\gcd(a,b)}$. So we have $ax + by = c$ where

$$x = x_0 \cdot \frac{c}{\gcd(a,b)} \quad \text{and} \quad y = y_0 \cdot \frac{c}{\gcd(a,b)}.$$

Example. Use the above approach to solve the LDE

$$66x + 15y = 18.$$

Solution. We'll start by using the Euclidean algorithm to find $\gcd(66, 15)$:

$$66 = 15(4) + 6 \quad \Rightarrow \quad \gcd(66, 15) = \gcd(15, 6) \qquad (1)$$

$$15 = 6(2) + 3 \quad \Rightarrow \quad \gcd(15, 6) = \gcd(6, 3) \qquad (2)$$

$$6 = 3(2) + 0 \quad \Rightarrow \quad \gcd(6, 3) = \gcd(3, 0) = \underline{3}.$$

Since gcd(66, 15) = 3 divides 18, there is a solution to this LDE. To find it, we work backwards through the Euclidean algorithm:

$$\begin{aligned} 3 &= 15 - \underline{6}(2) && \text{by (2)} \\ &= 15 - [66 - 15(4)](2) && \text{by (1)} \\ &= 15(9) + 66(-2) \end{aligned}$$

Thus, we arrive at the equation $66(-2) + 15(9) = 3$. A solution to our LDE can be obtained by multiplying the above expression by $18/3 = 6$:

$$66(-12) + 15(54) = 18.$$

1 Finding the Complete Solution to an LDE

Let's consider the following variation on the Mario example from last time:

Example (Goomba's revenge!)**.** Goomba has forgotten how to run, and instead can only jump forward or backward. He can make long jumps 6 metres in length, or short jumps 4 metres in length. Mario (who has also forgotten how to run) stands 16 metres away.

If Goomba makes x long jumps and y short jumps, find **all** possible choices of (x, y) that will allow him to land on Mario.

Just like last time, this problem can be represented as an linear Diophantine equation:

$$6x + 4y = 16.$$

By using the Euclidean algorithm and working backwards, we can find a particular solution:

$$6(8) + 4(-8) = 16.$$

So he will land on Mario by making 8 long jumps forward and 8 short jumps backward. But this is not the only way!

Provide at least 2 more solutions.

Uh oh... there may be quite a few solutions. How many more are there? How do we obtain them? To answer these questions, we're going to have to think cleverly!

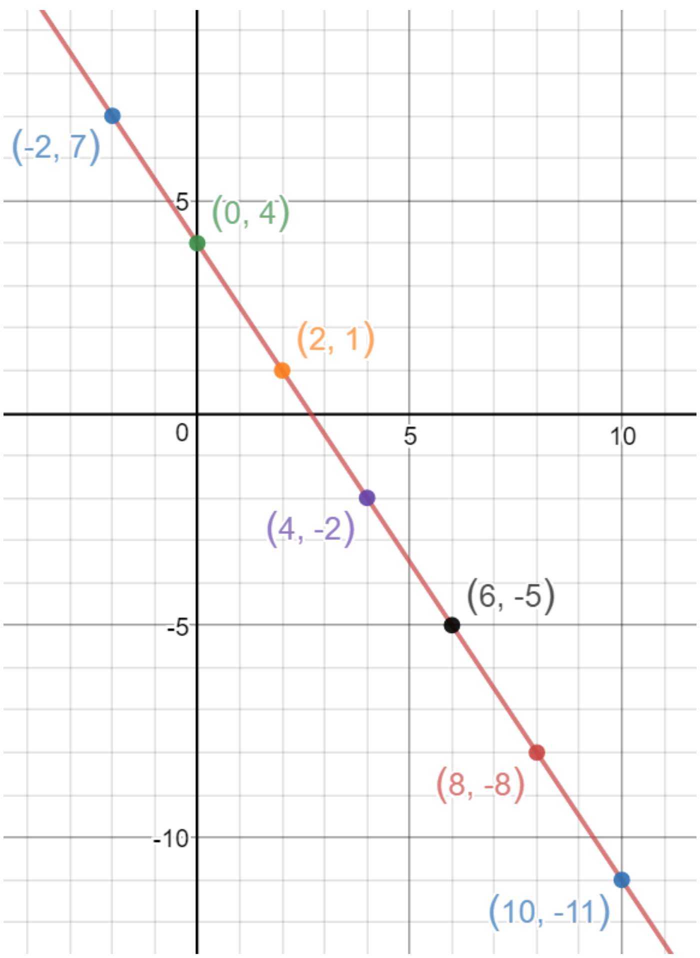
By moving things around a bit, the equation can be written as

$$6x + 4y = 16 \quad \longrightarrow \quad 4y = -6x + 16 \quad \longrightarrow \quad y = \frac{-6}{4}x + \frac{16}{4} \quad \longrightarrow \quad \boxed{y = \frac{-3}{2}x + 4}$$

We recognize this as the equation of a line with slope = __________

and y-intercept = __________.

The solutions to our LDE are exactly the points (x, y) on this line whose coordinates are integers!



On the left is the graph of the line $6x + 4y = 16$.

Below are some of the integer points on this line:

x	y	
$\vdots$	$\vdots$	
-2	7	
0	4	
2	1	
4	-2	
6	-5	
8	-8	$\longleftarrow$ our solution!
10	-11	
$\vdots$	$\vdots$	

These are the solutions to our LDE!

Do you notice a pattern?

Suppose we move along this line from top-left to bottom-right, starting at a particular solution (x_0, y_0).

To get to the next solution, we must add _____ to the x-value and subtract _____ from the y-value.

To get to the previous solution, we must subtract _____ from the x-value and add _____ to the y-value.

The following theorem summarizes this phenomenon:

Theorem. Suppose that (x_0, y_0) is one solution to the linear Diophantine equation $ax + by = c$, and let $d = \gcd(a, b)$. Then the full list of integer solutions is given by

$$x = x_0 + n\left(\frac{b}{d}\right), \quad y = y_0 - n\left(\frac{a}{d}\right)$$

where n is any integer.

Let's test out this theorem for $6x + 4y = 16$, the LDE from our previous example. We found that

- $a = 6, \, , b = 4,$
- $d = \gcd(6, 4) = 2$, and
- $(x_0, y_0) = (8, -8)$ is a particular solution.

This means that the full list of solutions is given by

$$\begin{aligned} x = x_0 + n\left(\frac{b}{d}\right) &= 8 + n\left(\frac{4}{2}\right) \\ &= \underline{8 + 2n} \end{aligned}$$

$$\begin{aligned} y = y_0 - n\left(\frac{a}{d}\right) &= -8 - n\left(\frac{6}{2}\right) \\ &= \underline{-8 - 3n} \end{aligned}$$

where n is any integer.

In the table below, we plug in various values for n. Each time we do, we get a different solution (x, y).

n	0	1	2	3	4	-1	-2	-3	-4	101	$\cdots$
$x = 8 + 2n$	8	10	_____	14	16	6	_____	2	0	_____	$\cdots$
$y = -8 - 3n$	-8	-11	-14	-17	_____	-5	-2	1	4	_____	$\cdots$

How many solutions does our LDE have? ______________________________.

Example. Find all integer solutions to the LDE

$$66x + 15y = 18.$$

Solution. This was the example from before. We saw that

- $a = 66$, $b = 15$,
- $d = \gcd(a, b) = 3$, and
- $(x_0, y_0) = (-12, 54)$ is a particular solution.

Thus, the full list of solutions is given by

$x =$
$y =$

2 Restrictions on Solutions to LDEs

Recall the following example from last time:

Example. Becky needs \$2.15 to buy an extra large coffee. She only has quarters and dimes, and the cashier insists that she pay with exact change.

Is there a combination of quarters and dimes that will total \$2.15?

Solution. This example is asking for a solution to the LDE

$$25x + 10y = 215.$$

By using the Euclidean algorithm and working backwards, we obtain the particular solution

$$25(43) + 10(-86) = 215.$$

It looks like Becky can buy her coffee with <u>43 quarters and −86 dimes.</u>

Wait... there's something wrong here: our solution doesn't make sense in the context of this problem. What we really want is a solution where both x and y are ________________ integers.

To accomplish this, we'll need to find *all* solutions to the LDE, and then restrict our attention to the ones that make sense. Since

- $a = 25$, $b = 10$,
- $d = \gcd(a, b) = 5$, and
- $(x_0, y_0) = (43, -86)$ is a particular solution,

so the full list of solutions is given by

$$\begin{aligned} x = x_0 + n\left(\frac{b}{d}\right) &= 43 + n\left(\frac{10}{5}\right) \\ &= \underline{43 + 2n} \end{aligned}$$

$$
\begin{aligned}
y = y_0 - n\left(\frac{a}{d}\right) &= -86 - n\left(\frac{25}{5}\right) \\
&= \underline{-86 - 5n}
\end{aligned}
$$

where n is any integer.

For (x, y) to be a valid solution, we need both $x \geq 0$ and $y \geq 0$. Which values of n will satisfy these conditions?

$$
\begin{aligned}
x \geq 0 \quad &\Rightarrow \quad 2n \geq -43 \\
&\Rightarrow \quad n \geq \frac{-43}{2} = -21.5 \\
&\Rightarrow \quad n \geq -21 \qquad \text{(since } n \text{ is an integer).}
\end{aligned}
$$

$$
\begin{aligned}
y \geq 0 \quad &\Rightarrow \quad 5n \leq -86 \\
&\Rightarrow \quad n \leq \frac{-86}{5} = -17.2 \\
&\Rightarrow \quad n \leq -18 \qquad \text{(since } n \text{ is an integer).}
\end{aligned}
$$

Below are the 4 valid solutions to this equation:

n	-18	-19	-20	-21
$x = 43 + 2n$	7	5	3	1
$y = -86 - 5n$	4	9	14	19

Remark. Solving for n in an inequality is just like solving for n in an equation. The only difference: we must reverse the inequality sign when multiplying or dividing by a negative number.