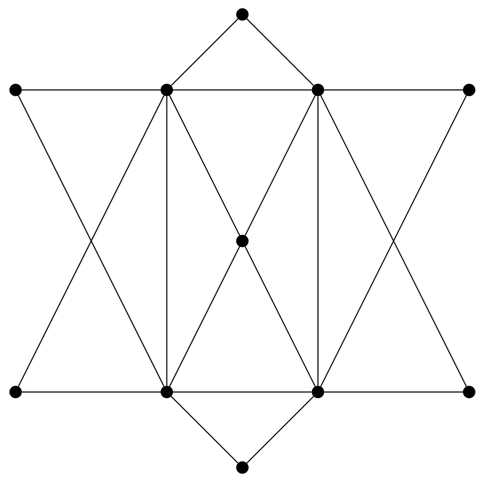
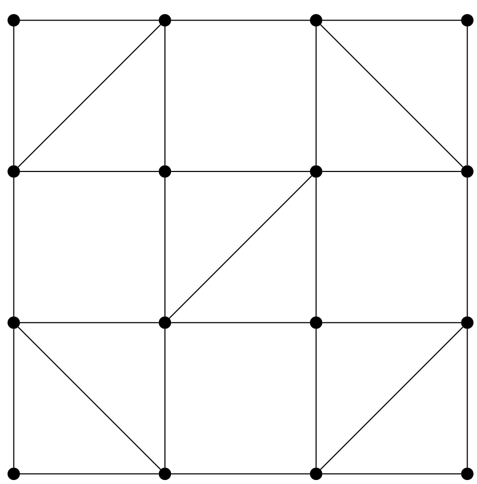
Intermediate Math Circles
Wednesday, February 22, 2017
Problem Set 3

1. For each graph below, determine whether it admits an Euler circuit, an Euler trail, or neither. If it has an Euler circuit or Euler trail, use Fleury's algorithm to find one.

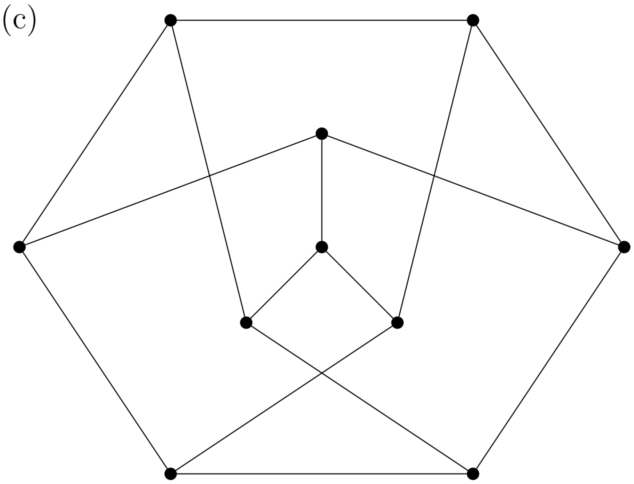
(a)



(b)

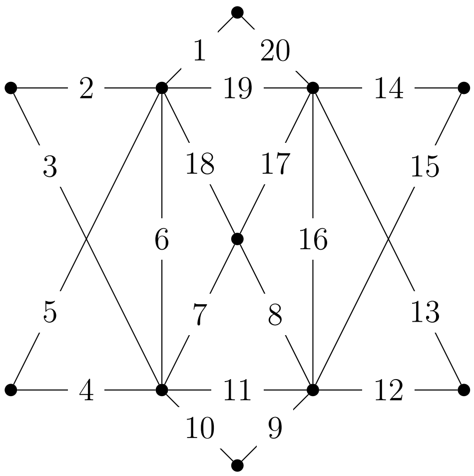


(c)

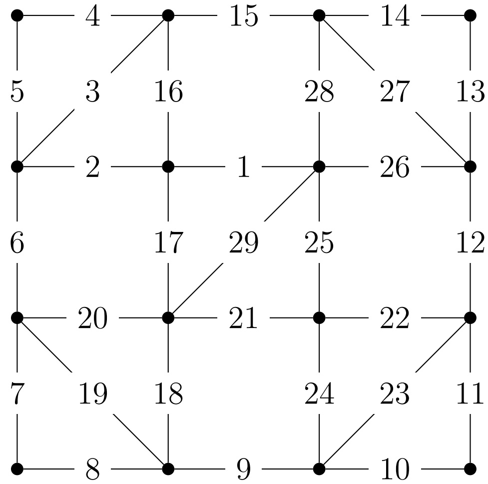


Solution:
Each vertex in (a) has even degree, and hence this graph has an Euler circuit. There are exactly two vertices in (b) that have odd degree, so this graph admits an Euler trail. Examples are provided below.

(a)



(b)



(c

There are more than two vertices in (c) with odd degree (in fact, the graph in (c) is 3-regular) and hence we have no chance of finding even an Euler trail.

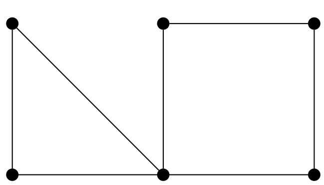
2. In each section below, find an example of a simple Eulerian graph G that satisfies the given conditions or explain why one doesn't exist.

   (a) G has an even number of vertices and an even number of edges.
   (b) G has an even number of vertices and an odd number of edges.
   (c) G has an odd number of vertices and an even number of edges.
   (d) G has an odd number of vertices and an odd number of edges.

Solution:

(a) The cycle graph on 4 vertices is 2-regular and hence is Eulerian. It has 4 vertices and 4 edges.

(b) The following graph is Eulerian as every vertex has even degree. It has 6 vertices and 7 edges.



(c) The complete graph on 5 vertices is 4-regular and hence is Eulerian. It has 5 vertices and 10 edges.

(d) The cycle graph on 3 vertices is 2-regular and hence is Eulerian. It has 3 vertices and 3 edges.

3. Let G be a connected graph that is not Eulerian. Show that we can introduce a single vertex to G along with some edges from the new vertex to the old ones so that the new graph is Eulerian.

Solution:
Firstly, note that since G is not Eulerian, there are some vertices in G of odd degree. Let $v_1, v_2, \ldots, v_k$ denote these vertices, and observe that k must be even as every graph has an even number of vertices of odd degree.

We'll introduce a new vertex v to our graph and connect it with a single edge to each of the vertices $v_1, v_2, \ldots, v_k$. It follows from the above remarks that $\deg(v) = k$ is even. Furthermore, since the vertices $v_1, v_2, \ldots, v_k$ had odd degrees initially, each now has even degree. Certainly the remaining vertices of G have even degree by construction. We've therefore enlarged G to a graph in which every vertex has even degree, and hence the resulting graph is Eulerian.

4. Let G be a connected simple graph with 14 vertices.

(a) If v is any vertex in G, what are the possible values of $\deg(v)$?

(b) Suppose that G has at most 7 vertices of degree 13. What is the greatest number of edges that G can have? **Hint:** Handshaking theorem.

(c) If G has 88 edges, explain why G cannot be Eulerian.

Solution:

(a) Since G is simple and connected $\deg(v)$ could be any integer between 1 and 13.

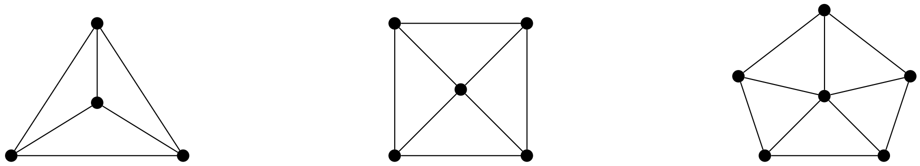
(b) Let m be the maximum number of edges in G. If G has at most 7 vertices of degree 13, then the maximum degree of each remaining vertex is 12. This means that the sum of the degrees in G is at most $7 \times 13 + (14 - 7) \times 12 = 175$. Since the sum of the degrees in G is twice the number of edges (by the Handshaking theorem), we have $2m \leq 175$. Dividing by 2, we see that

$$m \leq 175/2 = 87.5.$$

Of course, m must be an integer. We conclude that G has at most 87 edges.

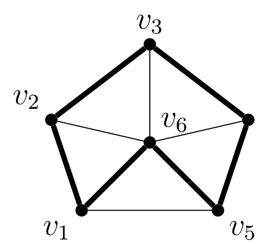
(c) If G has 88 edges, then G must have *at least* 8 vertices of degree 13 (for if it had 7 or fewer, the greatest number of edges possible would be 87 by the previous problem). This proves that G some vertices of odd degree, and hence cannot be Eulerian. Since it has more than 2 vertices of odd degree, we can't even hope to find an Euler trail in G.

5. Recall that for $n \geq 4$, the *wheel graph* W_n is made up of $n-1$ vertices connected in a cycle, with each of these vertices connected to an n^{th} vertex in the centre. For example, the wheel graphs W_4, W_5, and W_6 are drawn below.



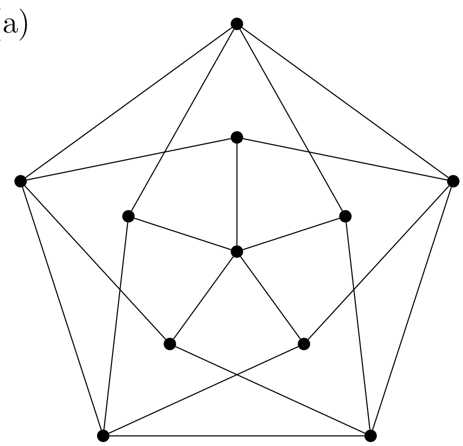
Explain how to find a Hamiltonian cycle in W_n for any $n \geq 4$.

Solution:
Start with any vertex on the outside. Moving clockwise from this vertex, label the outer vertices $v_1, v_2, \ldots, v_{n-1}$, and let v_n denote the vertex in the centre. To obtain a Hamiltonian cycle for W_n, simply follow the outer edges from v_1 to v_2, from v_2 to v_3, etc. until the vertex v_{n-1} is reached. Finish the cycle by moving from v_{n-1} to v_n, and then from v_n back to v_1. This technique is demonstrated on W_6 below.

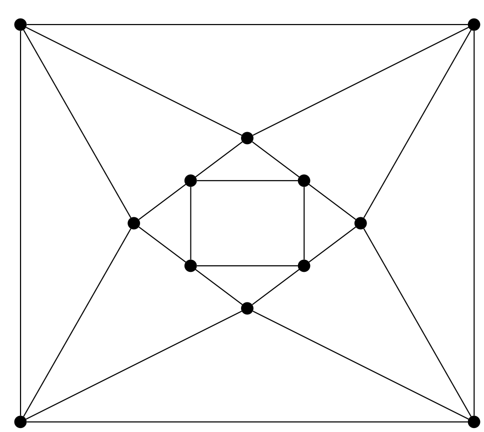


6. Find a Hamiltonian cycle in each of the following graphs.

(a)

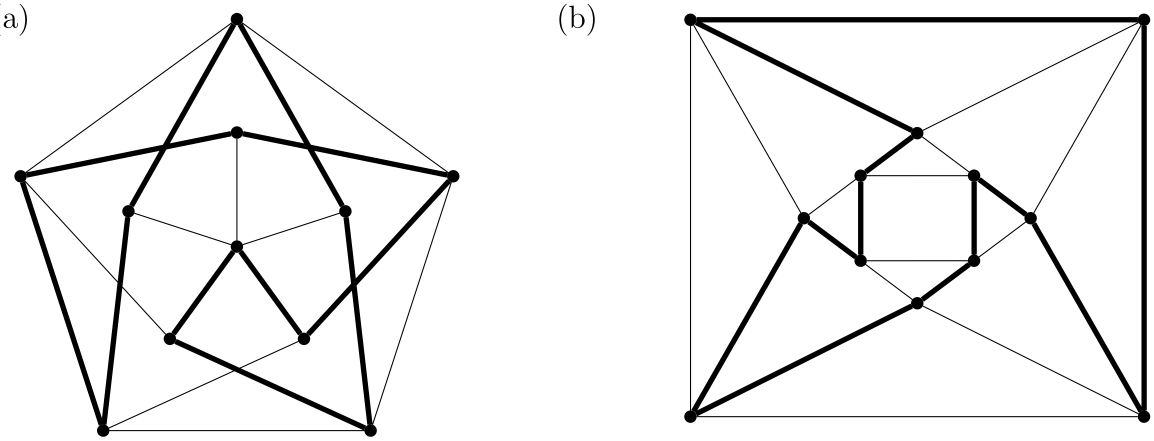


(b)



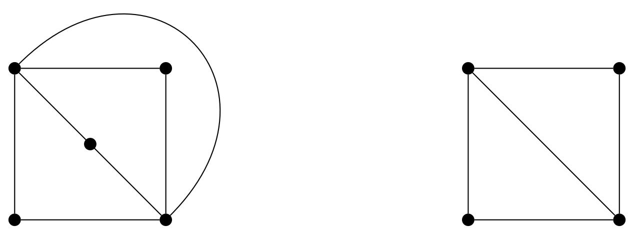
Solution:

Hamiltonian cycles are given by the thickened edges:



7. Is every Eulerian graph Hamiltonian? Is every Hamiltonian graph Eulerian? Justify your answers.

   Solution:
   It is not true that every Eulerian graph is Hamiltonian, nor is it true that every Hamiltonian graph is Eulerian. Consider the following graphs:

   The graph on the left is Eulerian but not Hamiltonian. Indeed, the degree of every vertex is even so it is Eulerian. To see that it is not Hamiltonian, observe that each edge incident to a vertex of degree 2 must be in any Hamiltonian cycle, which clearly cannot happen.

   It's easy to see that the graph on the right is Hamiltonian (a Hamiltonian cycle can be made using the outer edges) but the presence of vertices of odd degree shows it is not Eulerian.

8. Let G be a connected simple k-regular graph and suppose that G^c is connected.

   (a) Show that at least one of G or G^c is Eulerian.

   Solution:
   Let n be the number of vertices in G. Notice that if G is k-regular then every vertex in G^c is connected to $(n-1)-k$ vertices, and hence G^c is $(n-1)-k$ regular. We'll consider two cases.

(i) If k is even, then every vertex in G has even degree. Since G is connected, it follows that G is Eulerian.

(ii) Suppose instead that k is odd. If n were odd then G would have an odd number of vertices of odd degree, which is never the case. It must therefore be the case that n is even and hence $(n-1)-k$ is even as well. Since every vertex of G^c has degree $(n-1)-k$ (even) and G^c is connected (by assumption), we conclude that G^c is Eulerian.

(b) Show that at least one of G or G^c is Hamiltonian when n is even.[1] **Hint:** Dirac's theorem.

Solution:
Again, suppose that G has n vertices and note (as discussed in the previous solution) that G^c is $(n-1)-k$ regular. Recall the statement of Dirac's theorem: a simple graph with $n \geq 3$ vertices is Hamiltonian whenever $\deg(v) \geq n/2$. It's easy to see that the assumptions of the problem force our graph G to have at least 3 vertices.

With the above in mind, suppose that neither G nor G^c is Hamiltonian. Dirac's theorem states that G and G^c must each have a vertex of degree less than $n/2$, and hence $k < n/2$ and $(n-1)-k < n/2$. If n is even, then $n/2$ is an integer. This means that

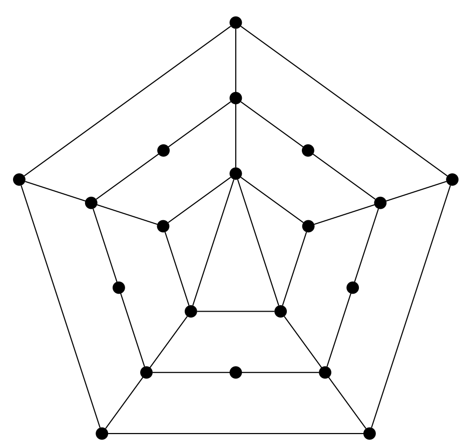
$$k \leq n/2 - 1$$

and

$$(n-1) - k \leq n/2 - 1.$$

Adding these inequalities we see that $n-1 \leq n-2$, which is clearly absurd. Since our reasoning led to a contradiction, we conclude that at least one of G or G^c must be Hamiltonian.

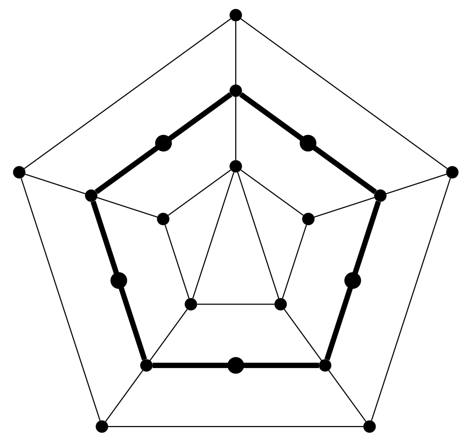
9. Find a Hamiltonian cycle in the following graph or explain why it doesn't exist.



Solution:
There are 5 vertices of degree 2 in this graph located on the middle pentagon. Observe

[1]In fact, this is true even when n is odd! Proving this case, however, is a bit more challenging.

that the edges incident to these vertices *must* be in any Hamiltonian cycle; there is no other way these vertices can be accessed. What this shows is that any Hamiltonian cycle must contain the thickened edges in our graph:



Of course, these edges already form a non-Hamiltonian cycle, and hence extending them to a Hamiltonian cycle is impossible. This proves that the graph is not Hamiltonian.