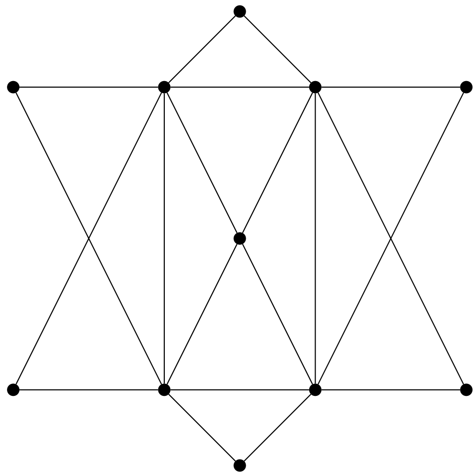
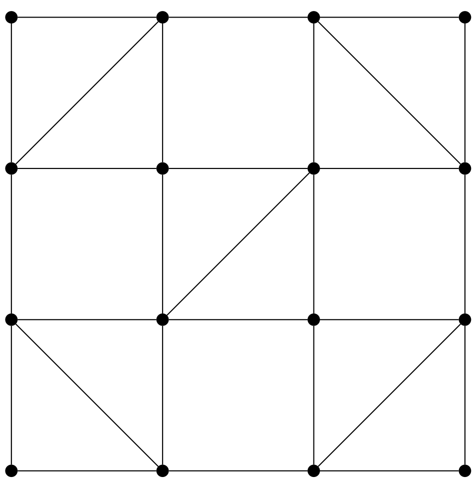
Intermediate Math Circles
Wednesday, February 22, 2017
Problem Set 3

1. For each graph below, determine whether it admits an Euler circuit, an Euler trail, or neither. If it has an Euler circuit or Euler trail, use Fleury's algorithm to find one.

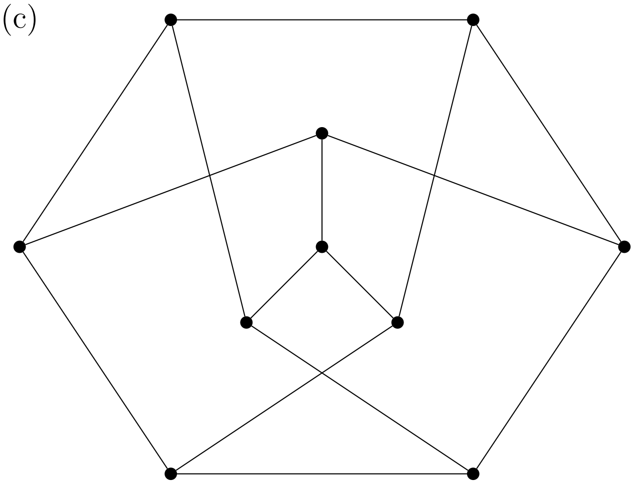
(a)



(b)



(c)



2. In each section below, find an example of a simple Eulerian graph G that satisfies the given conditions or explain why one doesn't exist.

   (a) G has an even number of vertices and an even number of edges.

   (b) G has an even number of vertices and an odd number of edges.

   (c) G has an odd number of vertices and an even number of edges.

   (d) G has an odd number of vertices and an odd number of edges.

3. Let G be a connected graph that is not Eulerian. Show that we can introduce a single vertex to G along with some edges from the new vertex to the old ones so that the new graph is Eulerian.

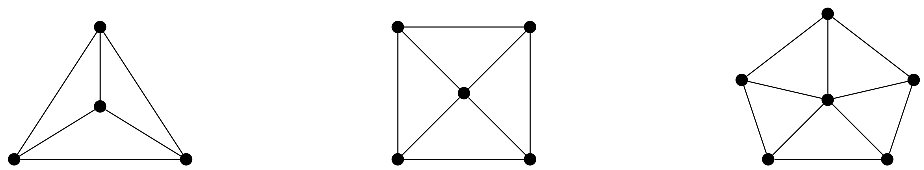
4. Let G be a connected simple graph with 14 vertices.

   (a) If v is any vertex in G, what are the possible values of $\deg(v)$?

   (b) Suppose that G has at most 7 vertices of degree 13. What is the greatest number of edges that G can have? **Hint:** Handshaking theorem.

   (c) If G has 88 edges, explain why G cannot be Eulerian.

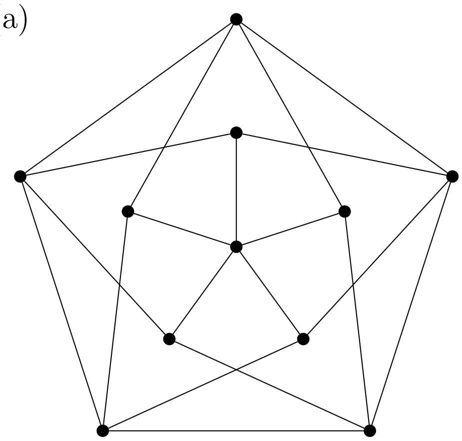
5. Recall that for $n \geq 4$, the *wheel graph* W_n is made up of $n-1$ vertices connected in a cycle, with each of these vertices connected to an n^{th} vertex in the center. For example, the wheel graphs W_4, W_5, and W_6 are drawn below.



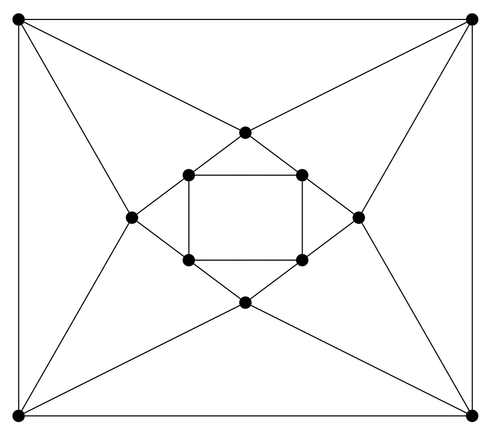
   Explain how to find a Hamiltonian cycle in W_n for any $n \geq 4$.

6. Find a Hamiltonian cycle in each of the following graphs.

(a)



(b)



7. Is every Eulerian graph Hamiltonian? Is every Hamiltonian graph Eulerian? Justify your answers.

8. Let G be a connected k-regular graph and suppose that G^c is connected.

   (a) Show that at least one of G or G^c is Eulerian.

   (b) Show that at least one of G or G^c is Hamiltonian when n is even.[1] **Hint:** Dirac's theorem.

[1] In fact, this is true even when n is odd! Proving this case, however, is a bit more challenging.

9. Find a Hamiltonian cycle in the following graph or explain why it doesn't exist.