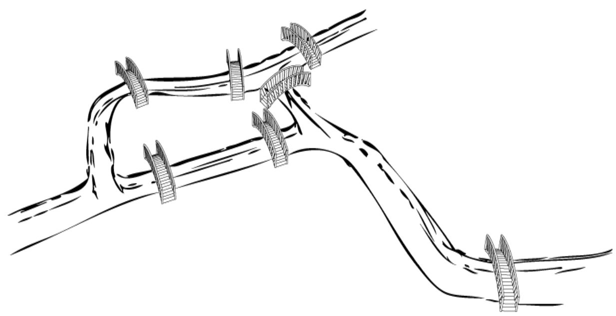
Intermediate Math Circles
Wednesday, February 22, 2017
Graph Theory III

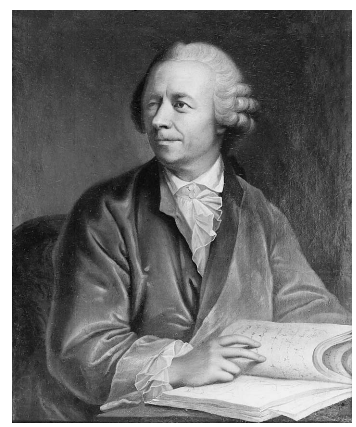
1 Eulerian Graphs

Let's begin this section with a problem that you may remember from lecture 1. Consider the layout of land and water given in the following map.



Is it possible to start at some point on the land and cross every bridge exactly once, ultimately returning to your starting position? This is known as the *Königsberg bridge problem.* Its name comes from Königsberg, Prussia (now Kaliningrad, Russia), a city whose layout in the 1700's resembled that of the above image.

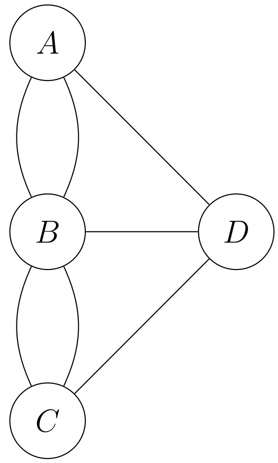
This question was answered in the negative by mathematician Leonhard Euler (right) in 1736. Euler's analysis of the problem used new mathematical arguments that laid the foundations of graph theory. What's more, Euler showed how to answer this question for *any* configuration of islands and bridges!



Leonhard Euler
1707-1783

Euler made a couple of key observations that helped to simplify the bridges problem. Firstly, the starting location is irrelevant; if there is a route that solves the problem, all landmasses must be visited and so it doesn't matter where one begins. Secondly, the movements that are made within a particular landmass are also unimportant.

Thus, the problem can be described using the following picture, where the circles A, B, C, and D represent the land masses and the connecting lines represent the bridges.



Euler's simplification of the bridge problem.

To solve the problem, Euler argued as follows: if there were a way to cross every bridge exactly once and return to the starting position, then any time a bridge is used to reach a particular island, there should be another (unused) bridge that allows us to leave. This means that the bridges connected to any island must come in pairs; one to enter and one to exit. Of course, this is not the case in our picture above; there are landmasses (all of them, in fact) that are connected to an odd number of bridges. Thus, Euler provided a negative resolution to the bridge problem.

1.1 Back to Graph Theory

It is likely the case that many of you noticed that Euler's simplified depiction of the bridges problem is really just a graph with vertices A, B, C, and D, and edges corresponding to the bridges. In order to state the bridges problem in a purely graph theoretic way, we will require a bit of terminology. Recall that a *walk* in a graph G is a route taken from one vertex to another by traversing the edges of G, and a *path* is a walk with no repeated vertices.

Definition. *Let G be a graph.*

(a) *A walk in G is said to be* ***closed*** *if it starts and ends at the same vertex.*

(b) *A* ***trail*** *in G is a walk with no repeated edges.*

(c) *A* ***circuit*** *in G is a closed walk with no repeated edges (i.e., a closed trail).*

Thus, the bridges of Königsberg problem is equivalent to asking whether or not the above graph admits a circuit that includes every edge. Such a circuit will be called an **Euler circuit**, and a graph that contains an Euler circuit will be called **Eulerian**. Euler's analysis showed that no graph containing vertices of odd degree can be Eulerian. Remarkably, this is essentially the only obstruction.

Theorem (Euler, 1736)**.** *A graph G is Eulerian if and only if it is connected and every vertex in G has even degree.*

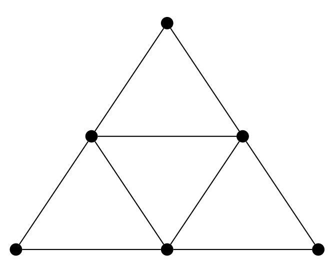
What if we're a bit less demanding? Can we find a route through Königsberg that crosses every bridge exactly once, possibly without returning to our point of departure? In our newly acquired graph theory language, this asks if there is a *trail* in the corresponding graph that includes every edge. Such a trail will be called an **Euler trail**.

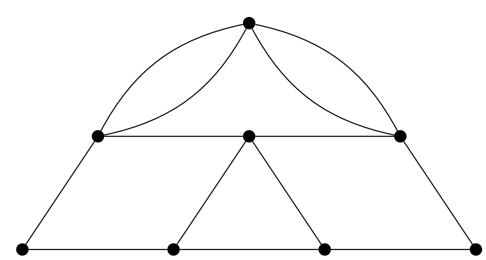
Suppose that G is a graph that has an Euler trail from vertex v to vertex w, but contains no Euler circuit. If we add an imaginary "ghost edge" from v to w, this trail can be extended to an Euler circuit. By the above result, every vertex in the new graph (i.e., with the ghost edge present) must have even degree. What changes when we remove the ghost edge? The degrees of v and w are each reduced by 1 (making them odd) and every other vertex is unaffected. Hence, any graph with an Euler trail but no Euler circuit has exactly two vertices of odd degree (acting as the starting and ending points of the trail). Conversely, we may argue in a similar manner to deduce that any connected graph with exactly 2 vertices of odd degree admits an Euler trail.

Corollary. *If G is a non-Eulerian graph, then G contains an Euler trail if and only if G is connected and has exactly 2 vertices of odd degree. The trail necessarily starts at one of these vertices and ends at the other.*

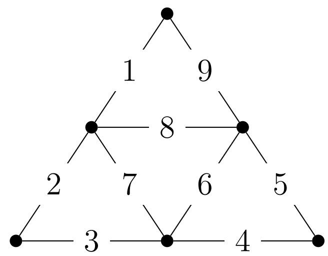
By appealing to this line of reasoning it is clear that the city of Königsberg contains no Euler trail; the corresponding graph has 4 vertices of odd degree. Can you find an Euler trail if an additional bridge is added from A to B? If we also add a bridge from C to D, can you find an Euler circuit?

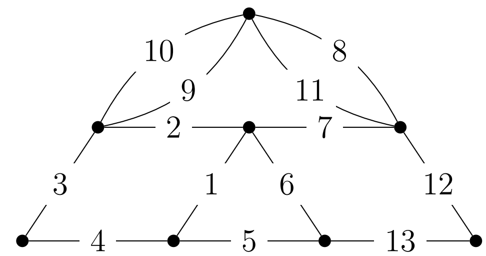
Let's see a couple more examples. The graph on the left is Eulerian while the one on the right contains only an Euler trail (note the two vertices of odd degree).



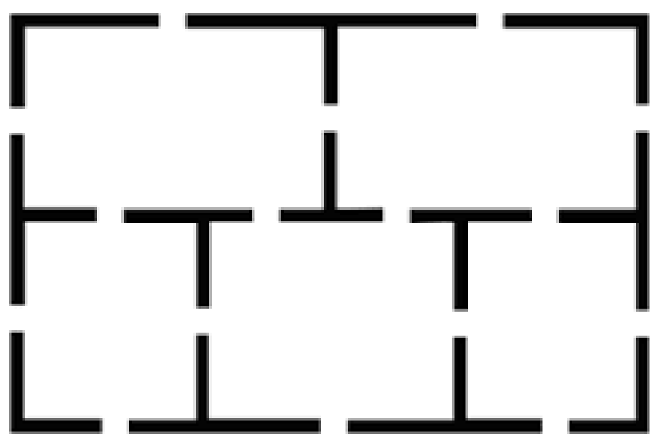


It's easy to find an Euler circuit or Euler trail once you know that one exists. Start at any vertex (if you're looking for an Euler trail, start at an odd-degree vertex). Choose any route you like, but cross an edge that disconnects the graph only when there are no other options. Remove edges as you go so that no edge is repeated. This process is known as *Fleury's algorithm.* Here we've used Fleury's algorithm to produce an Euler circuit and Euler trail for the graphs in the preceding example.





As an exercise, try to use Euler's results to solve the other problem posed in lecture 1. Namely, decide whether or not it is possible to find a route in the following figure that runs through each door exactly once and ends back at the starting point. If it is possible, find an example. Otherwise, explain why it cannot be done.



2 Hamiltonian Graphs

As we saw in the previous section, there is a straightforward method to decide whether or not a given graph G has an Euler circuit (i.e., a closed walk that visits every edge exactly once). It is natural to ask the same question about the *vertices* of G. That is, when does a graph G admit a closed walk that visits every vertex exactly once? This motivates the following definition.[1]

Definition. *Let G be a graph. A* ***cycle*** *in G is a closed walk with no repeated vertices (i.e., a closed path).*

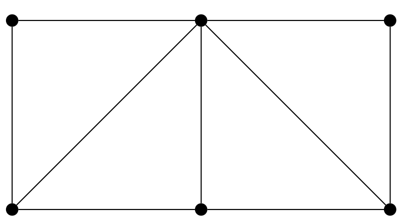
William Hamilton
1805-1865

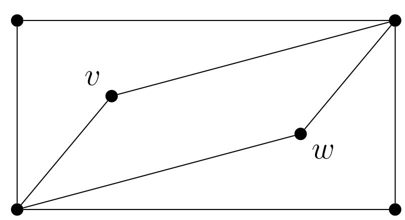
With this new definition in mind, the above question can be phrased as

"When does a graph contain
a cycle that visits every vertex?"

A cycle of this kind is called a **Hamiltonian cycle** in honour of mathematician William Hamilton (left) who, in 1857, invented a game that asked the player to find such a cycle on the vertices and edges of a dodecahedron. A graph that admits a Hamiltonian cycle is called a **Hamiltonian graph**. Surprisingly this question is *much* harder to resolve that its counterpart in section 1. There is currently no definitive answer to the problem.

For example, the graph on the left is Hamiltonian. An example of a Hamiltonian cycle is given by traversing the outside edges.





However, the graph on the right is *not* Hamiltonian. To see this, observe that every edge incident to a vertex of degree 2 necessarily belongs to any Hamiltonian cycle; we must be able to both visit and depart from any given vertex. This means that all 4 of the inside edges must belong to any Hamiltonian cycle, as v and w each have degree 2. But these edges form a non-Hamiltonian cycle already, and hence there is no cycle that includes all 6 vertices.
In general, the more edges in a graph, the better the chance it has at a Hamiltonian cycle. The following result due to Dirac is useful when the graph in question has lots of edges relative to its number of vertices.

[1] Does this definition make sense? Any closed walk *must* repeat a vertex (the vertex at the start/end). In a cycle we'll allow the start/end vertex to be visited twice, but no more than that!

Theorem (Dirac, 1952). *A simple graph G with $n \geq 3$ vertices is Hamiltonian if* $\deg(v) \geq n/2$ *for all vertices v in G.*

Evidently, the condition "$\deg(v) \geq n/2$ for all vertices v" is not *necesary* for a graph to exhibit a Hamiltonian cycle. Indeed, the graph on the left in the preceding example does not satisfy the conditions of Dirac's theorem, but does have a Hamiltonian cycle. An example where Dirac's theorem can be used is in the following graph. Try to find a Hamiltonian cycle explicitly.