# Grade 11/12 Math Circles
# Cardinality II - Problem Set

## March 30 - April 6, 2022
Zack Cramer

1. Answer the following questions involving the cardinality of power sets.
   (a) Let $A = \{a, b, c, d\}$. List the elements of $\mathcal{P}(A)$.
   (b) If $|\mathcal{P}(A)| = 8192$, what is $|A|$?
   (c) Does there exist a set $A$ such that $|\mathcal{P}(A)| = 100$? Explain.
   (d) If $|\mathcal{P}(\mathcal{P}(A))| = 2$, what can be said about $A$?
   (e) If $|\mathcal{P}(\mathcal{P}(A))|$ is less than 4 billion, what is the largest possible value of $|A|$?

*Solution*:

(a) There are $2^4 = 16$ elements in $\mathcal{P}(A)$. They are:

| | | | | | | |
|---|---|---|---|---|---|---|
| 0-element sets | $\emptyset$ | | | | | |
| 1-element sets | $\{a\}$ | $\{b\}$ | $\{c\}$ | $\{d\}$ | | |
| 2-element sets | $\{a,b\}$ | $\{a,c\}$ | $\{a,d\}$ | $\{b,c\}$ | $\{b,d\}$ | $\{c,d\}$ |
| 3-element sets | $\{a,b,c\}$ | $\{a,b,d\}$ | $\{a,c,d\}$ | $\{b,c,d\}$ | | |
| 4-element sets | $\{a,b,c,d\}$ | | | | | |

(b) Since $|\mathcal{P}(A)|$ is finite, $A$ must be finite. In particular, if $|A| = n$, then $|\mathcal{P}(A)| = 2^n$. So we must to determine the value of $n$ such that $2^n = 8192$. Trial and error will work, but if you know a bit about logarithms, you can also solve this using a calculator:

$$2^n = 8192 \implies \log_2(2^n) = \log_2(8192) \implies n = \log_2(8192) = 13.$$

Thus, $|A| = 13$.

(c) If $A$ is a finite set with cardinality $n$, then the cardinality of $|\mathcal{P}(A)|$ is $2^n$. But since

100 is not an integer power of 2, it must be the case that there does not exist a finite set $A$ with $|\mathcal{P}(A)| = 100$.

(d) If $|A| = n$, then $|\mathcal{P}(A)| = 2^n$, and hence $|\mathcal{P}(\mathcal{P}(A))| = 2^{(2^n)}$. Since we know that $|\mathcal{P}(\mathcal{P}(A))| = 2$, it must be the case that

$$2^{(2^n)} = 2 = 2^1.$$

By comparing exponents, we see that $2^n = 1$ and hence $n = 0$. This means that $|A| = 0$, so $A = \emptyset$.

Indeed, if $A = \emptyset$, then $\mathcal{P}(A) = \{\emptyset\}$ and $\mathcal{P}(\mathcal{P}(A)) = \{\emptyset, \{\emptyset\}\}$, so $|\mathcal{P}(\mathcal{P}(A))| = 2$.

(e) If $|A| = n$, then $|\mathcal{P}(A)| = 2^n$, and hence $|\mathcal{P}(\mathcal{P}(A))| = 2^{(2^n)}$. We must determine the largest value of $n$ such that $2^{(2^n)} < 4\,000\,000\,000$. Using a calculator, we compute a few values of $2^{(2^n)}$ and record the results in the table below:

| $n$ | $2^n$ | $2^{(2^n)}$ |
|---|---|---|
| 0 | 1 | 2 |
| 1 | 2 | 4 |
| 2 | 4 | 16 |
| 3 | 8 | 256 |
| 4 | 16 | 65 536 |
| 5 | 32 | 4 294 967 296 |

Even with just 5 elements in $A$, the set $\mathcal{P}(\mathcal{P}(A))$ has nearly 4.3 billion elements! Thus, the largest possible value for $|A|$ is $|A| = 4$.

2. Categorize the following sets based on their cardinality:

$$\mathbb{Z}, \quad \mathcal{P}(\mathbb{Z}_{11}), \quad \mathbb{R}\times\mathbb{R}\times\mathbb{R}, \quad \mathbb{Z}_2\times\mathbb{Z}_8, \quad \mathbb{N}\times\mathbb{Q}, \quad \mathcal{P}(\mathbb{N}), \quad \mathcal{P}(\mathbb{R}), \quad [0,1], \quad \mathcal{P}(\{0,1\}\times\{0,1\})$$

*Solution*:

- The set of integers, $\mathbb{Z}$, is countably infinite. We saw this in Lesson 1.
- Since $\mathbb{Z}_{11}$ has $n = 11$ elements, its power set, $\mathcal{P}(\mathbb{Z}_{11})$, will have $2^n = 2^{11} = 2048$ elements. In particular, this is a finite set.
- The Cartesian product $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ (also written as $\mathbb{R}^3$) was shown in the notes to have the same cardinality as $\mathbb{R}$.
- Since $|\mathbb{Z}_2| = 2$ and $|\mathbb{Z}_8| = 8$, there are $2 \cdot 8 = 16$ ways to form a pair $(a, b)$ with $a \in \mathbb{Z}_2$ and $b \in \mathbb{Z}_8$. Thus, $|\mathbb{Z}_2 \times \mathbb{Z}_8| = 16$.
- Since $\mathbb{N} \times \mathbb{Q}$ is a Cartesian product of two countable sets, this set is countable. More specifically, it is countably infinite.
- It was shown in the notes that $|\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$.
- By Cantor's Theorem, $|\mathcal{P}(\mathbb{R})|$ is even larger than $|\mathbb{R}|$.
- We saw in Lesson 1 that $|[0, 1]| = |\mathbb{R}|$.
- The set $\{0, 1\} \times \{0, 1\}$ consists of four pairs: $(0, 0)$, $(1, 0)$, $(0, 1)$, and $(1, 1)$. Thus, $\mathcal{P}(\{0, 1\} \times \{0, 1\})$ contains $2^4 = 16$ elements.

Having considered the cardinality of each set separately, we are now prepared to sort them into categories based on their size:

| $\lvert A\rvert = 16$ | $\lvert A\rvert = 2048$ | $\lvert A\rvert = \lvert\mathbb{N}\rvert$ | $\lvert A\rvert = \lvert\mathbb{R}\rvert$ | $\lvert A\rvert > \lvert\mathbb{R}\rvert$ |
|---|---|---|---|---|
| $\mathbb{Z}_2 \times \mathbb{Z}_8$ | $\mathcal{P}(\mathbb{Z}_{11})$ | $\mathbb{Z}$ | $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ | $\mathcal{P}(\mathbb{R})$ |
| $\mathcal{P}(\{0, 1\} \times \{0, 1\})$ | | $\mathbb{N} \times \mathbb{Q}$ | $\mathcal{P}(\mathbb{N})$ | |
| | | | $[0, 1]$ | |

3. Let $A$, $B$, $C$, $D$ be sets. Show that if $|A| = |C|$ and $|B| = |D|$, then $|A \times B| = |C \times D|$.

*Solution*: Since $|A| = |C|$, there exists a bijective function $f : A \to C$. Likewise, since $|B| = |D|$, there exists a bijective function $g : B \to D$.

To show that $|A \times B| = |C \times D|$, we must exhibit a bijective function $h : A \times B \to C \times D$.

To this end, consider the function

$$h : A \times B \to C \times D, \;\; h(a, b) = (f(a), g(b)).$$

To see that $h$ is surjective, let $(c, d)$ be an element of $C \times D$. Since $f : A \to C$ is surjective, there exists $a \in A$ such that $f(a) = c$; and likewise, since $g : B \to D$ is surjective, there exists $b \in B$ such that $g(b) = d$. Thus, we have

$$h(a, b) = (f(a), g(b)) = (c, d),$$

hence $h$ is surjective.

To see that $h$ is injective, consider two equal outputs $h(a_1, b_1) = h(a_2, b_2)$:

$$\begin{aligned} h(a_1, b_1) = h(a_2, b_2) &\implies (f(a_1), g(b_1)) = (f(a_2), g(b_2)) && \text{(by definition of } h\text{)} \\ &\implies f(a_1) = f(a_2) \text{ and } g(b_1) = g(b_2) \\ &\implies a_1 = a_2 \text{ and } b_1 = b_2 && \text{(since } f \text{ and } g \text{ are injective)} \\ &\implies (a_1, b_1) = (a_2, b_2). \end{aligned}$$

Since equal outputs must come from equal inputs, $h$ is injective. Thus, $h$ is bijective.

4. We have seen that a Cartesian products of finitely many countable sets is countable. That is, if $A_1, A_2, \ldots, A_n$ are countable, then so is $A_1 \times A_2 \times \cdots \times A_n$.

   Is the same true for a *countably infinite* collection of countable sets? That is, if $A_1, A_2, A_3, \ldots$ are countable sets, must $A_1 \times A_2 \times A_3 \times \cdots$ be countable as well?

   **Hint:** Let $A = \{0, 1, 2, \ldots, 9\}$. Is the Cartesian product $A \times A \times A \times \cdots$ countable? Think about decimal expansions.

*Solution*: As suggested in the hint, we will let $A = \{0, 1, 2, \ldots, 9\}$ and consider the Cartesian product $A \times A \times A \times \cdots$. This set contains all infinite tuples

$$(a_1, a_2, a_3, a_4, a_5, \ldots), \;\; \text{where} \;\; a_i \in \{0, 1, 2, \ldots, 9\}.$$

It will be helpful to change our perspective and think of these tuples as decimal numbers:

$$(a_1, a_2, a_3, a_4, a_5, \ldots) \longleftrightarrow 0.a_1\, a_2\, a_3\, a_4\, a_5\, \ldots$$

Since each $a_i$ can be any number from $\{0, 1, 2, \ldots, 9\}$, the tuples $(a_1, a_2, a_3, a_4, a_5, \ldots)$ can be used to construct any real number $0.a_1\, a_2\, a_3\, a_4\, a_4\, \ldots$ in $[0, 1]$. Since we know that $[0, 1]$ is uncountable, the set $A \times A \times A \times \cdots$ must be uncountable as well.

The moral here is that while a Cartesian product of *finitely* many countable sets is countable, the Cartesian product of *countably* many countable sets—even countably many *finite* sets—may be uncountable!

5. Prove that at any point $(a, b)$ in the $xy$-plane, there is a circle centred at $(a, b)$ that does not pass through any points of the form $(p, q)$ where $p$ and $q$ are rational.

   **Hint:** Compare the number of circles one can draw at an arbitrary point $(a, b)$ with the number of points $(p, q)$ where $p$ and $q$ are rational.

*Solution*: At each point $(a, b)$ in the $xy$-plane, there are uncountably many circles centred at $(a, b)$. Indeed, the equation of a circle centred at $(a, b)$ is

$$(x - a)^2 + (y - b)^2 = r^2,$$

where $r \in (0, \infty)$ is the radius of the circle. Since $(0, \infty)$ is an uncountable set, uncountably many such circles can be drawn.

Note, however, that there are only countably many points $(p, q)$ where $p$ and $q$ are rational. Indeed, these points are exactly the elements $\mathbb{Q} \times \mathbb{Q}$, which we know to be countable, since $\mathbb{Q}$ is countable. Thus, while there are uncountable many circles centred at $(a, b)$, only countably many of them can pass through points $(p, q)$ where $p$ and $q$ are rational.

Therefore, some (in fact, most) of the circles centred at $(a, b)$ must not pass through any rational points!

6. Recall from one of your earlier Math Circles lessons that a real number $\alpha$ is said to be **algebraic** if there is a polynomial

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where $a_0, a_1, \ldots, a_n$ are rational numbers, such that $p(\alpha) = 0$. For instance, $\sqrt{5}$ is algebraic, since $p(x) = x^2 - 5$ is a polynomial with rational coefficients and $p(\sqrt{5}) = 0$. If no such polynomial exists, $\alpha$ is said to be **transcendental.**

In this exercise, you will determine the cardinality of the set of algebraic numbers and the set of transcendental numbers.

(a) Let $\mathbb{P}_n$ be the set of all polynomials of degree $n$ with rational coefficients. For instance, $\mathbb{P}_2$ is the set of all polynomials of the form

$$p(x) = a_2 x^2 + a_1 x + a_0, \quad \text{where } a_0, a_1, a_2 \in \mathbb{Q}.$$

Show that $\mathbb{P}_n$ is countable by exhibiting a bijection

$$f : \mathbb{Q}^{n+1} \longrightarrow \mathbb{P}_n.$$

(b) Let $\mathbb{P}$ be the set of all polynomials with rational coefficients. Show that $\mathbb{P}$ is countable.

**Hint:** Proposition 1 from the notes.

(c) Let $\mathbb{A}$ denote the set of algebraic real numbers. Using part (b), as well as the fact that a polynomial of degree $n$ has at most $n$ real roots, show that $\mathbb{A}$ is countable.

(d) Let $\mathbb{T}$ denote the set of all transcendental real numbers. Show that $\mathbb{T}$ is uncountable.[1]

*Solution*:

(a) To see that $\mathbb{P}_n$ is countable, consider the function $f : \mathbb{Q}^{n+1} \longrightarrow \mathbb{P}_n$ defined by

$$f(a_0, a_1, \ldots, a_n) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

[1] This shows that although it's tough to write down specific examples of transcendental numbers, most real numbers are, in fact, transcendental!

One may verify that $f$ is both surjective and injective, and hence $f$ is a bijection. Therefore,

$$|\mathbb{Q}^{n+1}| = |\mathbb{P}_n|.$$

As a finite product of countable sets, $\mathbb{Q}^{n+1}$ is countable. Thus, so is $\mathbb{P}_n$.

(b) The set $\mathbb{P}$ can be thought of as a union of the sets $\mathbb{P}_n$:

$$\mathbb{P} = \bigcup_{n=1}^{\infty} \mathbb{P}_n = \mathbb{P}_1 \cup \mathbb{P}_2 \cup \mathbb{P}_3 \cup \cdots$$

Note that since each $\mathbb{P}_n$ was shown in (a) to be countable, $\mathbb{P}$ is a countable union of countable sets. Since we showed in our first live session that countable unions of countable sets are themselves countable, we conclude that $\mathbb{P}$ is countable, as claimed.

(c) Since $\mathbb{P}$ is countable, the polynomials in this set can be written as a list:

$$\mathbb{P} = \{p_1, p_2, p_3, p_4, \ldots\}.$$

For each $n$, let $R_n$ denote the set of roots of $p_n$. Since the algebraic numbers are exactly the roots of the polynomials in $\mathbb{P}$, we have

$$\mathbb{A} = \bigcup_{n=1}^{\infty} R_n = R_1 \cup R_2 \cup R_3 \cup \cdots$$

Note that since each polynomial has only finitely many roots, each set $R_n$ is finite, hence countable. Thus, $\mathbb{A}$ is a countable union of countable sets. Since we showed in our first live session that countable unions of countable sets are themselves countable, we conclude that $\mathbb{A}$ is countable, as claimed.

(d) Observe that every real number is either algebraic or transcendental. That is

$$\mathbb{R} = \mathbb{A} \cup \mathbb{T}.$$

It was shown in (c) that $\mathbb{A}$ is countable. If $\mathbb{T}$ were also countable, then $\mathbb{R} = \mathbb{A} \cup \mathbb{T}$ would be a union of two countable sets, hence countable. But since we know that $\mathbb{R}$ is, in fact, uncountable, it must be the case that $\mathbb{T}$ is uncountable as well.