Grade 11/12 Math Circles
Cardinality II - Problem Set

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1. Answer the following questions involving the cardinality of power sets.
   (a) Let $A = \{a, b, c, d\}$. List the elements of $\mathcal{P}(A)$.
   (b) If $|\mathcal{P}(A)| = 8192$, what is $|A|$?
   (c) Does there exist a set A such that $|\mathcal{P}(A)| = 100$? Explain.
   (d) If $|\mathcal{P}(\mathcal{P}(A))| = 2$, what can be said about A?
   (e) If $|\mathcal{P}(\mathcal{P}(A))|$ is less than 4 billion, what is the largest possible value of $|A|$?

2. Categorize the following sets based on their cardinality:

$$\mathbb{Z}, \quad \mathcal{P}(\mathbb{Z}_{11}), \quad \mathbb{R}\times\mathbb{R}\times\mathbb{R}, \quad \mathbb{Z}_2\times\mathbb{Z}_8, \quad \mathbb{N}\times\mathbb{Q}, \quad \mathcal{P}(\mathbb{N}), \quad \mathcal{P}(\mathbb{R}), \quad [0,1], \quad \mathcal{P}(\{0,1\}\times\{0,1\})$$

3. Let A, B, C, D be sets. Show that if $|A| = |C|$ and $|B| = |D|$, then $|A \times B| = |C \times D|$.

4. We have seen that a Cartesian product of finitely many countable sets is countable. That is, if $A_1, A_2, \ldots, A_n$ are countable, then so is $A_1 \times A_2 \times \cdots \times A_n$.

   Is the same true for a *countably infinite* collection of countable sets? That is, if $A_1, A_2, A_3, \ldots$ are countable sets, must $A_1 \times A_2 \times A_3 \times \cdots$ be countable as well?

   Hint: Let $A = \{0, 1, 2, \ldots, 9\}$. Is the Cartesian product $A \times A \times A \times \cdots$ countable? Think about decimal expansions.

5. Prove that at any point (a, b) in the xy-plane, there is a circle centred at (a, b) that does not pass through any points of the form (p, q) where p and q are rational.

   Hint: Compare the number of circles one can draw at an arbitrary point (a, b) with the number of points (p, q) where p and q are rational.

6. Recall from one of your earlier Math Circles lessons that a real number α is said to be **algebraic** if there is a polynomial

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where $a_0, a_1, \ldots, a_n$ are rational numbers, such that $p(\alpha) = 0$. For instance, $\sqrt{5}$ is algebraic, since $p(x) = x^2 - 5$ is a polynomial with rational coefficients and $p(\sqrt{5}) = 0$. If no such polynomial exists, α is said to be **transcendental.**

In this exercise, you will determine the cardinality of the set of algebraic numbers and the set of transcendental numbers.

(a) Let $\mathbb{P}_n$ be the set of all polynomials of degree n with rational coefficients. For instance, $\mathbb{P}_2$ is the set of all polynomials of the form

$$p(x) = a_2 x^2 + a_1 x + a_0, \quad \text{where } a_0, a_1, a_2 \in \mathbb{Q}.$$

Show that $\mathbb{P}_n$ is countable by exhibiting a bijection

$$f : \mathbb{Q}^{n+1} \longrightarrow \mathbb{P}_n.$$

(b) Let $\mathbb{P}$ be the set of all polynomials with rational coefficients. Show that $\mathbb{P}$ is countable.

Hint: Proposition 1 from the notes.

(c) Let $\mathbb{A}$ denote the set of algebraic real numbers. Using part (b), as well as the fact that a polynomial of degree n has at most n real roots, show that $\mathbb{A}$ is countable.

(d) Let $\mathbb{T}$ denote the set of all transcendental real numbers. Show that $\mathbb{T}$ is uncountable.[1]

[1] This shows that although it's tough to write down specific examples of transcendental numbers, most real numbers are, in fact, transcendental!