Grade 11/12 Math Circles
Cardinality I - Problem Set Solutions

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1. Which of the following functions are surjective? Which are injective?

   (a) $f : \mathbb{Z} \to \mathbb{N}, \;\; f(n) = |n| + 1$

   (b) $g : [0, \infty) \to [0, \infty), \;\; g(x) = \frac{x}{x+1}$

   (c) $h : \mathbb{R} \to \mathbb{R}, \;\; h(x) = \begin{cases} 2x & \text{if } x \text{ is rational,} \\ 3x & \text{if } x \text{ is irrational.} \end{cases}$

Solution:

(a) To show that f is surjective, we must verify that any $n \in \mathbb{N}$ can be produced as an output of f. Given $n \in \mathbb{N}$, we have that $n - 1$ is a non-negative element of $\mathbb{Z}$, and hence $|n - 1| = n - 1$. This means that

$$f(n - 1) = |n - 1| + 1 = (n - 1) + 1 = n.$$

Therefore, f is surjective.

Note that f is not injective, as $f(1) = |1| + 1 = 2$ and $f(-1) = |-1| + 1 = 2$, yet $1 \neq -1$.

(b) If g were surjective, it would mean that every $y \in [0, \infty)$ could be attained as an output of g. This cannot possibly be the case, however: since every $x \in [0, \infty)$ satisfies $x < x + 1$, we have $\frac{x}{x+1} < 1$. Thus, we will never be able to output values larger than 1, so g is *not* surjective.

[**Note:** The input x that produces a given output $y \in [0, \infty)$ is

$$x = \frac{y}{1 - y}.$$

This quantity, however, isn't defined when $y = 1$ and is negative (i.e., not in the domain) when $y > 1$.]

To see that g is injective, consider two equal outputs, $g(x_1) = g(x_2)$. We have

$$\begin{aligned} g(x_1) = g(x_2) &\implies \frac{x_1}{x_1+1} = \frac{x_2}{x_2+1} \\ &\implies x_1(x_2+1) = x_2(x_1+1) && \text{(cross multiplying)} \\ &\implies x_1x_2 + x_1 = x_1x_2 + x_2 && \text{(expanding)} \\ &\implies x_1 = x_2 && \text{(cancelling common terms)} \end{aligned}$$

Since we have just shown that equal outputs must come from equal inputs, g is injective.

(c) To see that h is surjective, let y be some element of $\mathbb{R}$, the codomain. We will consider two cases:

- If y is rational, then so too is $x = \frac{y}{2}$. Hence

$$h\left(\frac{y}{2}\right) = 2\left(\frac{y}{2}\right) = y.$$

- If y is irrational, then so too is $x = \frac{y}{3}$ (for if $\frac{y}{3}$ were instead rational, then $3\left(\frac{y}{3}\right) = y$ would also be rational, which we know is not the case.) Hence

$$h\left(\frac{y}{3}\right) = 3\left(\frac{y}{3}\right) = y.$$

The above argument shows that any rational or irrational number (i.e., any real number) can be produced as an output of h. Therefore, h is surjective.

To see that h is injective, suppose we have two equal outputs, $h(x_1) = h(x_2)$. Since h produces rational outputs from rational inputs and irrational outputs from irrational inputs, it must be the case that either x_1 and x_2 are both rational, or x_1 and x_2 are both irrational. We consider these cases separately:

- If x_1 and x_2 are rational, then

$$h(x_1) = h(x_2) \implies 2x_1 = 2x_2 \implies x_1 = x_2.$$

- If x_1 and x_2 are irrational, then

$$h(x_1) = h(x_2) \implies 3x_1 = 3x_2 \implies x_1 = x_2.$$

In either case, we see that equal outputs must come from equal inputs, hence h is injective.

2. For each pair of sets A and B shown below, prove that $|A| = |B|$ by finding a bijection $f : A \to B$.

   (a) $A = \{3n : n \in \mathbb{N}\} = \{3, 6, 9, 12, \ldots\}$
   $B = \{4m : m \in \mathbb{N}\} = \{4, 8, 12, 16, \ldots\}$

   (b) $A = \{2n : n \in \mathbb{Z}\} = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$
   $B = \{m^2 : m \in \mathbb{Z}\} = \{0, 1, 4, 9, \ldots\}$

   (c) $A = [0, 100]$
   $B = [0, 1]$

   (d) $A = (0, 1)$
   $B = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

   (e) $A = \mathbb{R}$
   $B = (0, \infty)$

Solution:

(a) Consider the matching of elements of A to elements of B shown below:

$$\begin{array}{c|ccccc} A & 3 & 6 & 9 & 12 & \cdots \\ & \downarrow & \downarrow & \downarrow & \downarrow & \cdots \\ B & 4 & 8 & 12 & 16 & \cdots \end{array}$$

This relationship is given by the function

$$f : A \to B,\ f(3n) = 4n.$$

This function is surjective, for if $4m$ is a typical element of B, then $3m \in A$ and $f(3m) = 4m$. To see that f is injective, consider two equal outputs, $f(3n_1) = f(3n_2)$. We have

$$f(3n_1) = f(3n_2) \implies 4n_1 = 4n_2 \implies n_1 = n_2.$$

Thus, f is bijective, and hence $|A| = |B|$.

(b) Our strategy will be to send the non-negative even integers to the even perfect squares and the negative even integers to the odd perfect squares. This relationship can be described by the following diagram:

$$\begin{array}{c|cccccccc} A & 0 & -2 & 2 & -4 & 4 & -6 & 6 & \cdots \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \cdots \\ B & 0 & 1 & 4 & 9 & 16 & 25 & 36 & \cdots \end{array}$$

If you'd like an explicit formula for this function, consider

$$f : A \to B,\ \ f(2n) = \begin{cases} (2n)^2 & \text{if } 2n \geq 0, \\ (2n+1)^2 & \text{if } 2n < 0. \end{cases}$$

One can show as in (a) that f is both surjective and injective, though the reasoning is a bit more involved. Hopefully the diagram presented above will convince you that f is indeed bijective.

(c) This part is interesting, as the interval $[0, 100]$ appears to be *much* larger than the interval $[0, 1]$. However, consider the function

$$f : A \to B,\ f(x) = \frac{x}{100}.$$

To see that f is surjective, let y be an element of $[0, 1]$. Since $0 \leq y \leq 1$, we have

$100 \cdot 0 \leq 100 \cdot y \leq 100 \cdot 1$ and hence $100y \in [0, 100]$. Furthermore,

$$f(100y) = \frac{100y}{100} = y.$$

Thus, f is indeed surjective. It's also not hard to see that f is injective:

$$f(x_1) = f(x_2) \implies \frac{x_1}{100} = \frac{x_2}{100} \implies x_1 = x_2.$$

Thus, f is bijective, and hence $|A| = |B|$.

(d) The strategy here will be similar to the strategy for (c), but we will need to both stretch and shift our domain. Consider the function

$$f : A \to B,\ f(x) = \pi x - \frac{\pi}{2}.$$

The idea here is that multiplying by π will stretch the interval $(0, 1)$ into the interval $(0, \pi)$; and then subtracting $\pi/2$ will shift the interval $(0, \pi)$ to the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$.

To see that f is surjective, consider a value y in $(-\frac{\pi}{2}, \frac{\pi}{2})$ and let $x = \frac{y+\frac{\pi}{2}}{\pi}$. Note that

$$\begin{aligned} -\frac{\pi}{2} < y < \frac{\pi}{2} &\implies 0 < y + \frac{\pi}{2} < \pi && \text{(adding } \tfrac{\pi}{2} \text{ to all sides)} \\ &\implies 0 < \frac{y + \frac{\pi}{2}}{\pi} < 1 && \text{(dividing by } \pi) \\ &\implies 0 < x < 1. \end{aligned}$$

So $x \in (0, 1)$, and we have

$$f(x) = \pi \left(\frac{y + \frac{\pi}{2}}{\pi} \right) - \frac{\pi}{2} = \left(y + \frac{\pi}{2} \right) - \frac{\pi}{2} = y.$$

Thus, f is surjective. The argument for injectivity is very similar to the argument in (c). We conclude that f is bijective, and therefore $|A| = |B|$.

(e) It turns out that almost any exponential function $f(x) = a^x$ will work here. We'll consider the function

$$f : A \to B,\ f(x) = e^x.$$

Since $e^x > 0$ for all x, this function does indeed output values in $(0, \infty)$. To see that it is surjective, note that if $y \in (0, \infty)$, then $x = \ln(y)$ is a real number and $f(x) = e^{\ln(y)} = y$. To see that f is injective, note that if we have equal outputs $f(x_1) = f(x_2)$, then

$$f(x_1) = f(x_2) \implies e^{x_1} = e^{x_2} \implies \ln(e^{x_1}) = \ln(e^{x_2}) \implies x_1 = x_2.$$

Thus, f is bijective, hence $|A| = |B|$.

3. [Exercise 1 from the notes] Let's return to the example of the Hilbert Hotel, where we begin with every room occupied by a guest. Suppose that a countably infinite number of buses arrive, each carrying a countably infinite numbers of guests. Devise a strategy to rearrange the guests in the Hilbert Hotel to make room for all these new guests, or argue that this is impossible.

Hint: Think about how we showed that the set of rational numbers, $\mathbb{Q}$, is countable.

Solution: Enumerate the existing guests as G_{01}, G_{02}, G_{03}, G_{04}, ... Next, if we assign natural numbers to each of our infinitely many buses and enumerate the guests in bus n as G_{n1}, G_{n2}, G_{n3}, G_{n4}, ..., we can arrange all of the guests in an infinite table, just as we did when exploring the cardinality of $\mathbb{Q}_+$:

$$\begin{array}{lcccccc} & G_{01} & G_{02} & G_{03} & G_{04} & G_{05} & \cdots \\ \text{Bus } 1 & G_{11} & G_{12} & G_{13} & G_{14} & G_{15} & \cdots \\ \text{Bus } 2 & G_{21} & G_{22} & G_{23} & G_{24} & G_{25} & \cdots \\ \text{Bus } 3 & G_{31} & G_{32} & G_{33} & G_{34} & G_{35} & \cdots \\ \text{Bus } 4 & G_{41} & G_{42} & G_{43} & G_{44} & G_{45} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

We can now use the same strategy as when listing the elements of $\mathbb{Q}_+$: we can thread through this table diagonally, starting at the top left entry. Note that unlike in the example with $\mathbb{Q}_+$, this table has no repetition (as all of our guests are distinct people).

	G_{01}	G_{02}	G_{03}	G_{04}	G_{05}	$\cdots$
Bus 1	G_{11}	G_{12}	G_{13}	G_{14}	G_{15}	$\cdots$
Bus 2	G_{21}	G_{22}	G_{23}	G_{24}	G_{25}	$\cdots$
Bus 3	G_{31}	G_{32}	G_{33}	G_{34}	G_{35}	$\cdots$
Bus 4	G_{41}	G_{42}	G_{43}	G_{44}	G_{45}	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

This gives us the following one-to-one correspondence between the guests and the rooms in the hotel:

Guests	G_{01}	G_{02}	G_{11}	G_{21}	G_{12}	G_{03}	G_{04}	G_{13}	G_{22}	G_{31}	$\cdots$
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\cdots$
Rooms	1	2	3	4	5	6	7	8	9	10	$\cdots$

4. Let A and B be sets. We say that A is a **subset** of B, and write $A \subseteq B$, if every element of A is also an element of B. For instance, $\{0,2\} \subseteq \{0,1,2,3\}$ since $0 \in \{0,1,2,3\}$ and $2 \in \{0,1,2,3\}$. Another example:

$$\mathbb{Z}_n \subseteq \mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}.$$

(a) Fill in the blanks using the words **countable** and **uncountable** to make each statement true. You can mix-and-match the words as needed and use each word multiple times. Afterwards, explain why your completed statement is true.

i. If $A \subseteq B$ and A is ________________________, then B is ________________________.

ii. If $A \subseteq B$ and B is ________________________, then A is ________________________.

(b) Show that the set of all prime numbers is countable. (In fact, it is countably infinite.)

(c) Show that the set of real numbers, $\mathbb{R}$, is uncountable.

Solution:

(a) For (i), the correct statement should read:

If $A \subseteq B$ and A is **uncountable**, then B is **uncountable**.

The rationale is that if A is uncountable, then there are so many elements in this set that they cannot all possibly be contained in any infinite list. But note that any infinite list of the elements of B—the larger set—would contain within it a list of the elements of A, which we've assumed cannot exist. Therefore, the elements of B also cannot be expressed in an infinite list, hence B is uncountable.

For (ii), the correct statement should read:

If $A \subseteq B$ and B is **countable**, then A is **countable**.

The rationale is that if B—the larger set—is countable, then its elements can be written in a (possibly infinite) list. Within this list we will find a list of the elements of A. Since the elements of A can be listed, A is countable.

(b) Note that every prime number is a positive integer, hence the set of prime numbers is a subset of $\mathbb{N}$. Since $\mathbb{N}$ is countable, by part a(ii) above, so too is the set of all prime numbers.

(c) Note that $(0, 1)$ is a subset of $\mathbb{R}$. Since we have shown that $(0, 1)$ is uncountable, by part a(i) above, so too is $\mathbb{R}$.

5. Let A, B, and C be sets. Show that if $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.

Note: Does this look obvious to you? If so, remember that we're not really dealing with our usual notion of equality for real numbers — $|A|$, $|B|$, and $|C|$ represent (possibly infinite) cardinalities. To get started, think about what it means for two sets to have the same cardinality. What exactly needs to be shown here?

Solution: Recall that "$|A| = |B|$" means that there exists is bijection $f : A \to B$. So we need to show that if there exist bijections $f : A \to B$ and $g : B \to C$, then there also exists a bijection $h : A \to C$.

Let's assume we have bijections $f : A \to B$ and $g : B \to C$. Consider the function

$$h : A \to C, \;\; h(x) = g(f(x)).$$

(This function is known as the **composition** of f and g, and is often written as $g \circ f$.) We claim that h is a bijection.

To see that h is surjective, let c be an element of C. Since $g : B \to C$ is a bijection and hence surjective, there exists some $b \in B$ such that $g(b) = c$. Likewise, since $f : A \to B$ is bijective and hence surjective, there exists some $a \in A$ such that $f(a) = b$. Therefore,

$$h(a) = g(f(a)) = g(b) = c.$$

Since any $c \in C$ can be obtained as an output of h, the function h is indeed surjective.

To see that h is injective, suppose we have equal outputs $h(a_1) = h(a_2)$. We will show that $a_1 = a_2$. Indeed,

$$\begin{aligned} h(a_1) = h(a_2) &\implies g(f(a_1)) = g(f(a_2)) \\ &\implies f(a_1) = f(a_2) && \text{(since } g \text{ is bijective, hence injective)} \\ &\implies a_1 = a_2 && \text{(since } f \text{ is bijective, hence injective)} \end{aligned}$$

Thus, h is injective, and this proves that h is a bijection.

Consequently, if $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$, as required.

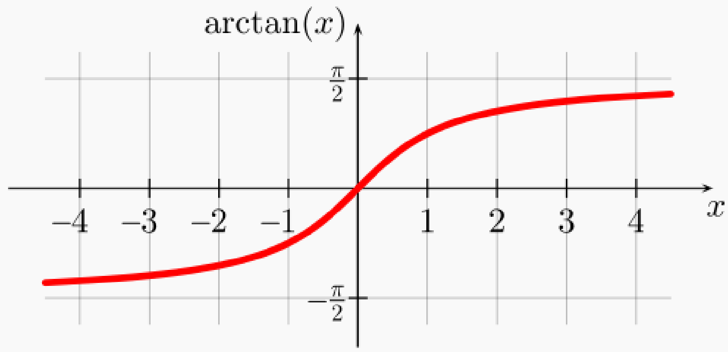
6. Show that $|\mathbb{R}| = |(0, 1)|$.

Hint: There are a few different ways to do this. One approach I can think of involves a trig function and some of the other problems on this worksheet.

Solution: Consider the function

$$f : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \quad f(x) = \arctan(x).$$

(Perhaps you've seen this function written as $f(x) = \tan^{-1}(x)$ – these are two different ways of describing the same function!) The graph of f is shown below:



This function accepts real numbers as inputs and can output any value in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. The function is therefore surjective. For injectivity, note that for any given output (y-value) in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, there is only one input (x-value) in $\mathbb{R}$ that will produce this output. This can be verified more precisely: since $\tan(x)$ and $\arctan(x)$ are inverses of each other, we have

$$\begin{aligned} f(x_1) = f(x_2) &\implies \arctan(x_1) = \arctan(x_2) \\ &\implies \tan(\arctan(x_1)) = \tan(\arctan(x_2)) \\ &\implies x_1 = x_2. \end{aligned}$$

Thus, f is injective.

We've just argued that $f : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is a bijection, and therefore $|\mathbb{R}| = \left|\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right|$. By problem 2(d), we also know that $\left|\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right| = |(0,1)|$. Thus, since $|\mathbb{R}| = \left|\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right|$ and $\left|\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right| = |(0,1)|$, problem 5 tells us that $|\mathbb{R}| = |(0,1)|$, as desired.

Alternatively, if you'd like an explicit bijection $f : \mathbb{R} \to (0,1)$, consider the function

$$f(x) = \frac{\arctan(x)}{\pi} + \frac{1}{2}.$$

(Think about how I might have come up with this function!)

7. If you've seen the 2014 film *The Fault in Our Stars* or read the 2012 John Green novel of the same name, you may have encountered the following curious quote on the concept of infinity:

> *"There are infinite numbers between 0 and 1. There's .1 and .12 and .112 and an infinite collection of others. Of course, there is a bigger infinite set of numbers between 0 and 2, or between 0 and a million. Some infinities are bigger than other infinities."*

Analyze this statement using what we've learned about cardinality. What part of this statement is correct? What part of this statement is incorrect? Explain.

Solution: We have seen that different sizes of infinity do exist. For example, the cardinality of $(0, 1)$ describes a larger infinity than the cardinality of $\mathbb{N}$. So the statement , *"Some infinities are bigger than other infinities"* is true.

What's not true, however, is the claim that "*there is a bigger infinite set of numbers between 0 and 2 or between 0 and a million*" than between 0 and 1. We showed in problem 2(c) that the intervals $[0, 1]$ and $[0, 100]$ have the same cardinality, and very similar arguments can be used to show that

$$|[0, 1]| = |[0, 2]| = |[0, 1\,000\,000]|.$$