Grade 11/12 Math Circles
Cardinality I - Problem Set

March 9-23, 2022

Zack Cramer

1. Which of the following functions are surjective? Which are injective?

   (a) $f : \mathbb{Z} \to \mathbb{N}, \;\; f(n) = |n| + 1$

   (b) $g : [0, \infty) \to [0, \infty), \;\; g(x) = \frac{x}{x+1}$

   (c) $h : \mathbb{R} \to \mathbb{R}, \;\; h(x) = \begin{cases} 2x & \text{if } x \text{ is rational,} \\ 3x & \text{if } x \text{ is irrational.} \end{cases}$

2. For each pair of sets A and B shown below, prove that $|A| = |B|$ by finding a bijection $f : A \to B$.

   (a) $A = \{3n : n \in \mathbb{N}\} = \{3, 6, 9, 12, \ldots\}$
   $B = \{4m : m \in \mathbb{N}\} = \{4, 8, 12, 16, \ldots\}$

   (b) $A = \{2n : n \in \mathbb{Z}\} = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$
   $B = \{m^2 : m \in \mathbb{Z}\} = \{0, 1, 4, 9, \ldots\}$

   (c) $A = [0, 100]$
   $B = [0, 1]$

   (d) $A = (0, 1)$
   $B = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

   (e) $A = \mathbb{R}$
   $B = (0, \infty)$

3. [Exercise 1 from the notes] Let's return to the example of the Hilbert Hotel, where we begin with every room occupied by a guest. Suppose that a countably infinite number of buses arrive, each carrying a countably infinite numbers of guests. Devise a strategy to rearrange the guests in the Hilbert Hotel to make room for all these new guests, or argue that this is impossible.

   Hint: Think about how we showed that the set of rational numbers, $\mathbb{Q}$, is countable.

4. Let A and B be sets. We say that A is a **subset** of B, and write $A \subseteq B$, if every element of A is also an element of B. For instance, $\{0, 2\} \subseteq \{0, 1, 2, 3\}$ since $0 \in \{0, 1, 2, 3\}$ and $2 \in \{0, 1, 2, 3\}$. Another example:

$$\mathbb{Z}_n \subseteq \mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}.$$

   (a) Fill in the blanks using the words **countable** and **uncountable** to make each statement true. You can mix-and-match the words as needed and use each word multiple times. Afterwards, explain why your completed statement is true.

      i. If $A \subseteq B$ and A is ________________________, then B is ________________________.

      ii. If $A \subseteq B$ and B is ________________________, then A is ________________________.

   (b) Show that the set of all prime numbers is countable. (In fact, it is countably infinite.)

   (c) Show that the set of real numbers, $\mathbb{R}$, is uncountable.

5. Let A, B, and C be sets. Show that if $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.

   Note: Does this look obvious to you? If so, remember that we're not really dealing with our usual notion of equality for real numbers — $|A|$, $|B|$, and $|C|$ represent (possibily infinite) cardinalities. To get started, think about what it means for two sets to have the same cardinality. What exactly needs to be shown here?

6. Show that $|\mathbb{R}| = |(0, 1)|$.

   Hint: There are a few different ways to do this. One approach I can think of involves a trig function and some of the other problems on this worksheet.

7. If you've seen the 2014 film *The Fault in Our Stars* or read the 2012 John Green novel of the same name, you may have encountered the following curious quote on the concept of infinity:

   "There are infinite numbers between 0 and 1. There's .1 and .12 and .112 and an infinite collection of others. Of course, there is a bigger infinite set of numbers between 0 and 2, or between 0 and a million. Some infinities are bigger than other infinities."

   Analyze this statement using what we've learned about cardinality. What part of this statement is correct? What part of this statement is incorrect? Explain.