# Grade 11/12 Math Circles
# Cardinality I

## March 9-23, 2022

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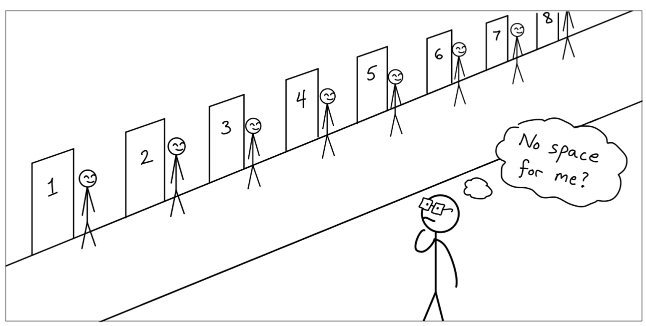
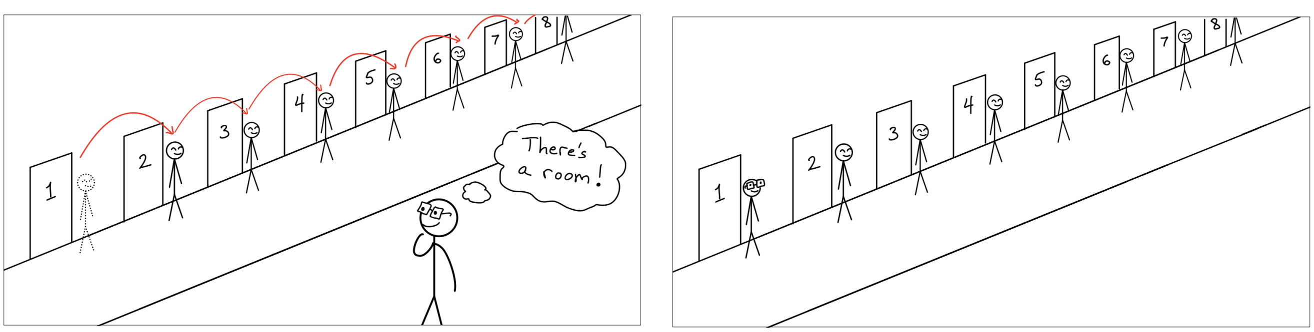
The *cardinality* of a set of objects is a measure of the set's size, meaning the number of objects in that set. This topic is interesting, beautiful, and in some cases, very counter-intuitive! To get you warmed up, consider the following scenario.

### The Hilbert Hotel

Imagine you're the manager of a hotel with infinitely many rooms, each labelled with a positive integer. Each room can accommodate just a single guest, and we'll assume that we begin with every room currently occupied. Suppose that a new guest arrives and is looking for a room.



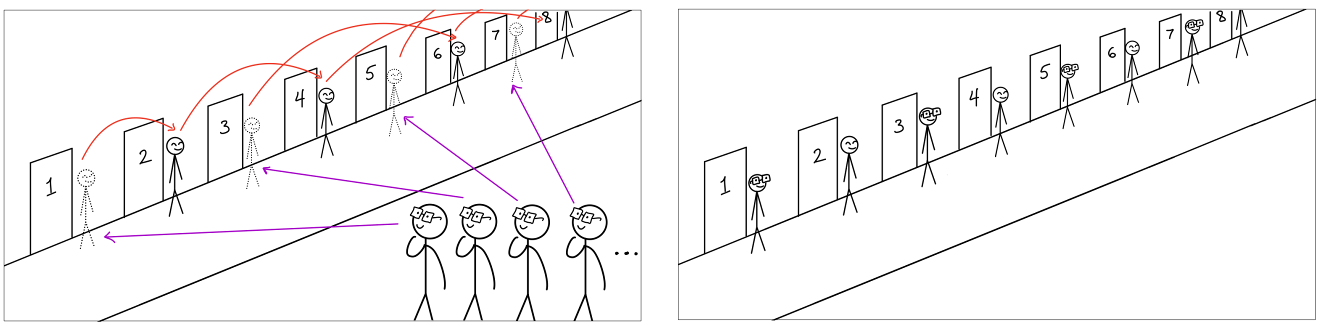

At first it may seem that there is no room for this guest – there appear to be more guests than there are rooms! Suddenly, however, you have an idea: if every guest moves to the room with number one higher than their current room number, every guest will have a place to stay, and Room 1 will become free for the new guest!



By moving the guest in Room 1 to Room 2, the guest in Room 2 to Room 3, and in general, the guest in Room $n$ to Room $(n+1)$, each guest (including the new one!) can be matched to a room. So maybe there *aren't* more guests than rooms after all!

A similar strategy can be used to make room for 2, 3, 4, or any other *finite* number of new guests. But what if a bus arrives with *infinitely* many new guests: New Guest 1, New Guest 2, New Guest 3, etc.? Is there a way to find room for all of them?

One way to do this is by moving each existing guest to the room with room number equal to double the number of their current room. That is, the guest in Room 1 will move to Room 2, the guest in Room 2 will move to Room 4, the guest in Room 3 will move to Room 6, and so forth. This will leave all the odd-numbered rooms vacant! We can then assign New Guest 1 to Room 1, New Guest 2 to Room 3, New Guest 3 to Room 5, etc., thereby providing everyone with a room.



**Exercise 1:**
What if infinitely many buses arrive, each carrying infinitely many guests? Devise a strategy to rearrange the guests in the Hilbert Hotel to make room for all these new guests, or argue that it's impossible to do so.

The Hilbert Hotel problem was a thought experiment proposed by mathematician David Hilbert in 1924 to illustrate the counter-intuitive properties of infinite sets. We'll return to this problem later on in our lessons.

David Hilbert: 1862 - 1943

# Sets

Since we'll be interested in discussing the *cardinality* (or *size)* of a set, we should first define (at least in a loose sense) what a set is.

**Definition 1:**
A **set** is a collection of distinct objects. Each object that appears in this collection is called an **element** of the set.

I'll note here that above definition of a set is far from precise: What is a "collection"? What are "certain objects"? Mathematicians have a more formal way of defining sets, but this is beyond the scope of our discussion. Rather than dwell on the formalities, let's explore some examples.

We'll primarily be interested in sets of numbers, but sets can include objects that are far more diverse. We can describe sets by listing their elements explicitly between curly braces, such as the set $S = \{2, 3, 5, 7\}$ whose elements are the prime numbers less than 10. We may write $2 \in S$ to indicate that 2 is an element of $S$. Likewise, we may write $1 \notin S$ to indicate that 1 is *not* an element of $S$.

The ordering of elements within a set does not matter: for example, the sets

$$A = \{\text{Pippin}, \text{Marco}, \text{Carly}\} \quad \text{and} \quad B = \{\text{Carly}, \text{Pippin}, \text{Marco}\}$$

are considered the same, even though the elements are presented in different orders. Since the sets have exactly the same elements—in this case, the names of my cats—we write $A = B$.

Below are more examples of sets of that arise very often in mathematics and will be of particular interest to us.

**Special Examples of Sets**

1. The **empty set**, $\emptyset$, consisting of no elements at all!
2. The set of **natural numbers**, $\mathbb{N} = \{1, 2, 3, \ldots\}$.
3. The set of **integers**, $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.
4. The set of natural numbers up to $n$, $\mathbb{Z}_n = \{1, 2, 3, \ldots, n\}$. For example, $\mathbb{Z}_4 = \{1, 2, 3, 4\}$.

5. The set of **rational numbers**,

$$\mathbb{Q} = \left\{ \frac{a}{b} : a, b \in \mathbb{Z} \text{ and } b \neq 0 \right\}.$$

   The notation tells us that the elements of $\mathbb{Q}$ have the form $\frac{a}{b}$ where $a$ and $b$ satisfy the conditions following the colon. In this case, $a$ and $b$ are required to be integers with $b \neq 0$. So $\mathbb{Q}$ is the set of all fractions of integers.

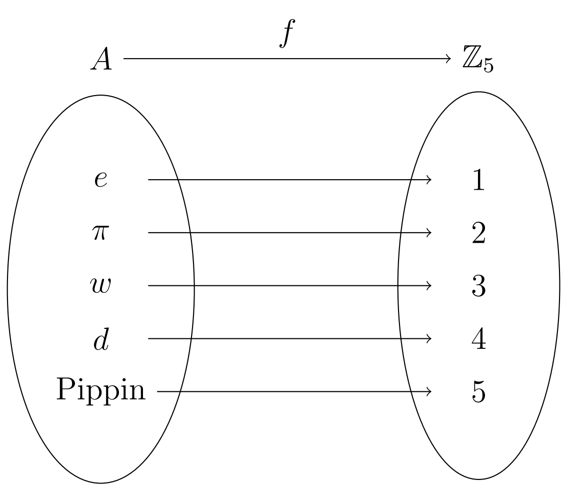
6. The set of **real numbers**, $\mathbb{R}$, consists of all numbers in decimal form.

Finally, note that our definition of a set requires the elements of a set to be distinct. That is, repetition of elements is not allowed. So, for example, $\{1, 1\}$ would not be considered a set.

## Cardinality

How many elements are in the set $A = \{e, \pi, w, d, \text{Pippin}\}$?

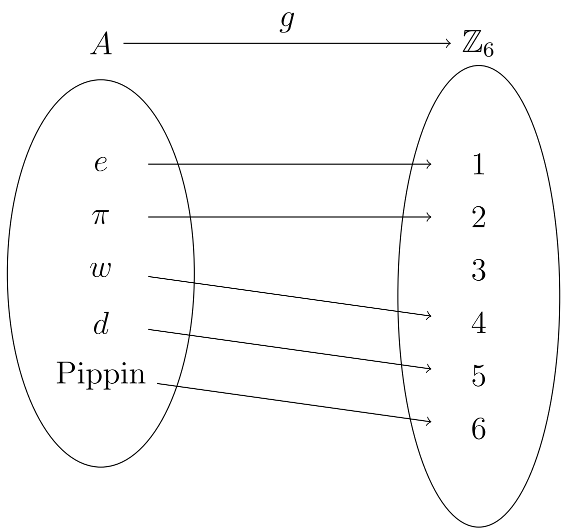
You probably said "five" without much thought. If you had to explain your process, maybe you would say that you moved through the set $A$ from left to right, assigning distinct numbers to the elements in the set: $e$ is number 1, $\pi$ is number 2, $a$ is number 3, and so on. In this way, you have defined a **function**, $f$, from the set $A$ to the set $\mathbb{Z}_5$ that "counts" the number of elements in $A$:




$$f(e) = 1$$
$$f(\pi) = 2$$
$$f(w) = 3$$
$$f(d) = 4$$
$$f(\text{Pippin}) = 5$$

The notation $f : A \to \mathbb{Z}_5$ is a common way to indicate that the inputs of the function $f$ are elements of $A$ and the outputs of $f$ are elements of $\mathbb{Z}_5$. We call $A$ the **domain** of $f$, while $\mathbb{Z}_5$ is called the **codomain** of $f$.

Note that we could have defined other functions on the set $A$, but not all of these functions will do as good of a job as $f$ when it comes to "counting" the elements of the domain. For instance, consider a function $g : A \to \mathbb{Z}_6$ defined as follows:




$g(e) = 1$

$g(\pi) = 2$

$g(w) = 4$

$g(d) = 5$

$g(\text{Pippin}) = 6$

Since we were able to define a function $g : A \to \mathbb{Z}_6$, does this mean that the set $A$ actually has *six* elements? Well... no. This function $g$ isn't really "counting" the elements of $A$ using the numbers $1, 2, 3, 4, 5, 6$, as no element of $A$ was assigned to the number 3.

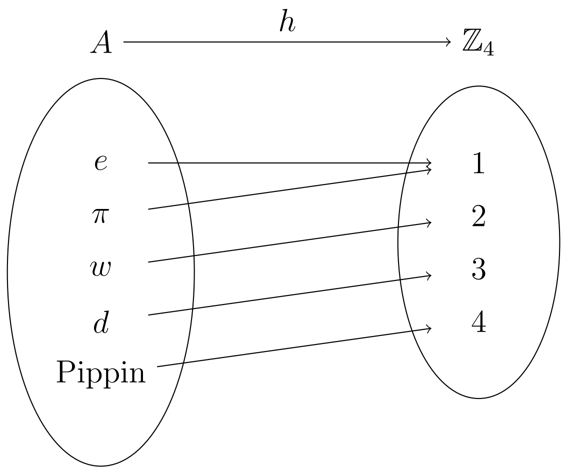
The takeaway here is that any function $f : A \to \mathbb{Z}_n$ that is meant to count the elements of $A$ must make use of *all* elements of the codomain. That is, we need to be able to produce *every* element of $\mathbb{Z}_n$ as an output of $f$. Such functions are said to be **onto** or **surjective**.

**Definition 2:**
Let $A$ and $B$ be sets. A function $f : A \to B$ is said to be **onto** or **surjective** if the following property holds:

For every element $b \in B$, there exists some element $a \in A$ such that $f(a) = b$.

The function $g$ in the previous example is not surjective, since 3 is an element of the codomain, but there is no element $a$ in the domain with $g(a) = 3$.

Let's explore another issue that can arise when trying to count the elements of a set using functions from $A$ to $\mathbb{Z}_n$. Consider a function $h : A \to \mathbb{Z}_4$ defined as follows:




$h(e) = 1$

$h(\pi) = 1$

$h(w) = 2$

$h(d) = 3$

$h(\text{Pippin}) = 4$

Since we were able to define a function $h : A \to \mathbb{Z}_4$, does this mean that $A$ actually has *four* elements? Once again, the answer is no. The problem here is that since *multiple* elements of $A$ are being assigned to a particular number (in this case, 1) in $\mathbb{Z}_4$, the function $h$ is not properly counting the elements of $A$ using the numbers $1, 2, 3, 4$.

The takeaway here is that any function $f : A \to \mathbb{Z}_n$ that is meant to count the elements of $A$ must have the property that distinct elements in domain are assigned distinct elements in the codomain. Such functions are said to be **one-to-one** or **injective**.

**Definition 3:**

Let $A$ and $B$ be sets. A function $f : A \to B$ is said to be **one-to-one** or **injective** if the following property holds:

For all elements $a_1 \in A$ and $a_2 \in A$, if $f(a_1) = f(a_2)$, then $a_1 = a_2$.

The function $h$ in the previous example is *not* injective since $h(e) = h(\pi)$, but $e \neq \pi$.

Let's examine these properties in some additional examples.

**Example 1**

Which of the functions below are surjective? Which are injective?

(a) $f : \mathbb{N} \to \{0, 1\}$, $f(n) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$

(b) $g : \mathbb{R} \to \mathbb{R}$, $g(x) = \sqrt{x^2 + 4}$

(c) $h : \mathbb{R} \to \mathbb{R}$, $h(x) = 4x + 3$.

*Solution:*

(a) The function $f$ is surjective, since we can achieve every element of $\{0, 1\}$ as an output of $f$. For instance, $f(2) = 0$ and $f(5) = 1$. The function is *not* injective, however, since $f(2) = f(4) = 0$, but $2 \neq 4$.

(b) Note that since square roots are always non-negative, we have $g(x) = \sqrt{x^2 + 4} \geq 0$ for all $x$. This means that $-1$, for example, could not be produced as an output of $g$, and therefore $g$ is *not* surjective. Note as well that $g$ is *not* injective, since $g(1) = g(-1) = \sqrt{5}$ but $1 \neq -1$.

(c) To show that $h$ is surjective, we must show that every $y \in \mathbb{R}$ can be achieved as an output of $h$. Let's think... given $y \in \mathbb{R}$, what would we need to input for $x$ in order to have $h(x) = y$?

$$\begin{aligned} h(x) = y \quad &\Longrightarrow \quad 4x + 3 = y \\ &\Longrightarrow \quad 4x = y - 3 \\ &\Longrightarrow \quad x = \tfrac{y-3}{4}. \end{aligned}$$

Sure enough, if we consider the input $x = \frac{y-3}{4}$, we do indeed get

$$h(x) = h\left(\frac{y-3}{4}\right) = 4\left(\frac{y-3}{4}\right) + 3 = (y-3) + 3 = y.$$

Since any $y \in \mathbb{R}$ can be achieved as an output of $h$, we conclude that $h$ is surjective.

To check whether $h$ is injective, suppose that there are elements $x_1, x_2 \in \mathbb{R}$ such that $h(x_1) = h(x_2)$:

$$\begin{aligned} h(x_1) = h(x_2) \quad &\Longrightarrow \quad 4x_1 + 3 = 4x_2 + 3 \\ &\Longrightarrow \quad 4x_1 = 4x_2 \quad \text{(subtracting 3)} \\ &\Longrightarrow \quad x_1 = x_2 \qquad \text{(dividing by 4)}. \end{aligned}$$

We have just argued that the only way for $h$ to output the same values from inputs $x_1$ and $x_2$ is if $x_1 = x_2$. Thus, we conclude that $h$ is injective.

**Exercise 2:**

We saw that the function $g$ in Example 1(b) was not surjective since there are elements of the codomain that cannot be obtained as outputs of $g$. Show that if we change codomain from $\mathbb{R}$ to $[2, \infty)$, the function

$$g : \mathbb{R} \to [2, \infty), \; g(x) = \sqrt{x^2 + 4}$$

is now surjective.

**Exercise 3:**

Consider the function

$$p : \mathbb{Z} \to \mathbb{R}, \; p(n) = 4n + 3.$$

This looks very similar to the function $h$ in Exercise 1(c), but the domain has been changed from $\mathbb{R}$ to $\mathbb{Z}$.

Unlike $h$, the function $p$ is *not* surjective. Demonstrate that that this is the case by finding an element in the codomain of $p$ that cannot be obtained as an output of $p$. Where does the argument for the surjectivity of $h$ in Example 1(c) break down when applied to $p$?

**Definition 4:**

Let $A$ and $B$ be sets. If $f : A \to B$ is both surjective and injective, we say that $f$ is **bijective** or that $f$ is a **bijection.**

In Example 1, the function $h$ bijective, while $f$ and $g$ are not.

Why do we care about bijections? Well, if there exists a bijective function $f : A \to B$ between sets $A$ and $B$, then

- $f$ assigns a *distinct* element from $B$ to each element of $A$ (injectivity), and
- *every* of element of $B$ is assigned (surjectivity).

That is, $f$ matches the elements of $A$ with elements of $B$, suggesting that in some sense, the two sets have the same number of elements. Thus, bijections give us a way to compare the number of elements in two different sets!

In the special case that $B = \mathbb{Z}_n$:

- $f$ assigns a *distinct* number in $\{1, 2, \ldots, n\}$ to each element of $A$ (injectivity), and
- *every* number in $\{1, 2, \ldots, n\}$ is assigned (surjectivity).

Essentially, $f$ is "counting" the elements of $A$ by matching these elements with numbers in $\{1, 2, \ldots, n\}$.

**Definition 5:**

Let $A$ and $B$ be sets. If there exists a bijection $f : A \to B$, we say that $A$ and $B$ are in **one-to-one correspondence** or that $A$ and $B$ have the same **cardinality**. In this case, we write $|A| = |B|$.

If, for some positive integer $n$, there is a bijection $f : A \to \mathbb{Z}_n$, we write $|A| = |\mathbb{Z}_n| = n$. In this case, or when $A = \emptyset$, we say that $A$ is a **finite set**. A set that is not finite is said to be **infinite**.

Using our new terminology, the function $f : \{e, \pi, w, d, \text{Pippin}\} \to \mathbb{Z}_5$ given by

$$f(e) = 1, \quad f(\pi) = 2, \quad f(w) = 3, \quad f(d) = 4, \quad f(\text{Pippin}) = 5$$

is a bijection. Therefore, $\{e, \pi, w, d, \text{Pippin}\}$ is a finite set and $|\{e, \pi, w, d, \text{Pippin}\}| = 5$.

**Stop and Think**

The function $f : \{e, \pi, w, d, \text{Pippin}\} \to \mathbb{Z}_5$ is not the *only* bijection between these sets. For example, the function $g : \{e, \pi, w, d, \text{Pippin}\} \to \mathbb{Z}_5$ given by

$$g(e) = 4, \quad g(\pi) = 2, \quad g(w) = 1, \quad g(d) = 5, \quad g(\text{Pippin}) = 3$$

is also bijective. If $A$ is set with $|A| = n$, how many bijective functions $g : A \to \mathbb{Z}_n$ exist?

You may be thinking, "Zack, why did we go through all this trouble to formalize the idea of counting? It was obvious before discussing bijections that $\{e, \pi, w, d, \text{Pippin}\}$ had 5 elements!"

Perhaps these ideas do feel overly formal in the context of *finite* sets. But what about when dealing with sets of *infinitely many* elements? We saw in our Hilbert Hotel example that infinite sets can be quite counter-intuitive, and so we need mathematically-rigorous tools to work with these sets precisely. Bijective functions are exactly the tools we need!

# Infinite Sets

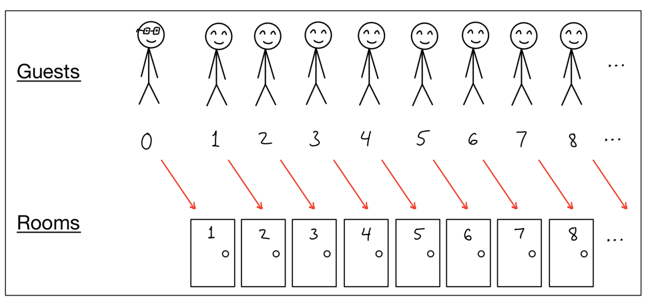
Let's return to our example of the Hilbert Hotel. We began by considering the problem of starting with every room occupied and attempting to make room for one additional guest.

We'll label the initial guest-room pairs using the elements of the set $\mathbb{N}$: Guest 1 is in Room 1, Guest 2 is in Room 2, Guest 3 is in Room 3, and so on. If the new guest is assigned the number 0, then matching all of the guests to rooms in the hotel is exactly the same as finding a one-to-one correspondence between the set $A = \{0, 1, 2, 3, \ldots\}$ (representing the guests) and the set $\mathbb{N} = \{1, 2, 3, \ldots\}$ (representing the rooms).

Sure enough, our strategy of shifting each guest one room to the right tells us that the function

$$f : A \to \mathbb{N}, \quad f(n) = n + 1$$

will be a bijection that establishes this one-to-one correspondence.




According to Definition 5, since there exists a bijective function $f : \{0, 1, 2, \ldots\} \to \mathbb{N}$, the sets $\{0, 1, 2, \ldots\}$ and $\mathbb{N}$ have the same cardinality: $|\{0, 1, 2, \ldots\}| = |\mathbb{N}|$.

It may seem a little bizarre that $\mathbb{N}$ has the same cardinality as $\{0, 1, 2, \ldots\}$, as the latter set appears to be strictly larger. But rest assured, no error has been made – this is just one of the curious situations that one can encounter when dealing with infinity!

**Definition 6:**

If $A$ is a set with $|A| = |\mathbb{N}|$, then $A$ is said to be **countably infinite**. Sets that are finite or countably infinite are said to be **countable**.

Countable sets are sets whose elements can be written as a (possibly infinite) list or sequence:

$$\begin{array}{c|ccccc} \mathbb{N} & 1 & 2 & 3 & 4 & \cdots \\ & \downarrow & \downarrow & \downarrow & \downarrow & \cdots \\ A & a_1 & a_2 & a_3 & a_4 & \cdots \end{array}$$

For this reason, I enjoy using the term **listable** as an alternative to *countable* (though the term "listable" is not widely used.)

**Example 2:**

The set of integers, $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, is countably infinite.

*Proof:* We need to show that the elements of $\mathbb{Z}$ can be written as an infinite list. One method is to weave the negative integers through the positive integers as follows:

$$0, 1, -1, 2, -2, 3, -3, 4, -4, \ldots$$

This establishes a one-to-one correspondence between the integers and the natural numbers, which can be described by the bijective function

$$f : \mathbb{Z} \to \mathbb{N}, \quad f(n) = \begin{cases} 2n & \text{if } n > 0, \\ 2|n| + 1 & \text{if } n \leq 0. \end{cases}$$

$\square$

In our second scenario of the Hilbert Hotel, we made room for infinitely many new guests by moving each existing guest from Room $n$ to Room $2n$ and slotting the new guests into the then vacant odd-numbered rooms. If we think of $\mathbb{N}$ as representing the set of infinitely many guests in the hotel and $\{0, -1, -2, \ldots\}$ as representing the set of infinitely many new guests looking for rooms, the function $f$ in Example 2 carries out this rearrangement process exactly!

**Example 3:**

The set of rational numbers, $\mathbb{Q}$, is countably infinite.

*Proof:* Just like in the last example, we must show that the elements of $\mathbb{Q}$ can be written as an infinite list. Let's start by showing that set of *positive* rational numbers, $\mathbb{Q}_+$, can be written as an infinite list.

We can write the elements of $\mathbb{Q}_+$ in an infinite table with the first row consisting of fractions with denominator 1, the second row consisting of fractions with denominator 2, etc.:

$$\begin{array}{cccccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \frac{2}{4} & \frac{2}{5} & \cdots \\
\frac{3}{1} & \frac{3}{2} & \frac{3}{3} & \frac{3}{4} & \frac{3}{5} & \cdots \\
\frac{4}{1} & \frac{4}{2} & \frac{4}{3} & \frac{4}{4} & \frac{4}{5} & \cdots \\
\frac{5}{1} & \frac{5}{2} & \frac{5}{3} & \frac{5}{4} & \frac{5}{5} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

Here's the fun part: we can write this table as a single list by starting in the top left corner and weaving through the table diagonally as shown below:

$$\begin{array}{cccccc}
\boxed{\frac{1}{1}} & \boxed{\frac{1}{2}} & \boxed{\frac{1}{3}} & \boxed{\frac{1}{4}} & \boxed{\frac{1}{5}} & \cdots \\
\boxed{\frac{2}{1}} & \frac{2}{2} & \boxed{\frac{2}{3}} & \frac{2}{4} & \frac{2}{5} & \cdots \\
\boxed{\frac{3}{1}} & \boxed{\frac{3}{2}} & \frac{3}{3} & \frac{3}{4} & \frac{3}{5} & \cdots \\
\boxed{\frac{4}{1}} & \frac{4}{2} & \frac{4}{3} & \frac{4}{4} & \frac{4}{5} & \cdots \\
\boxed{\frac{5}{1}} & \frac{5}{2} & \frac{5}{3} & \frac{5}{4} & \frac{5}{5} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

Notice that since our table contained repetition (e.g., $\frac{1}{1}$ is the same as $\frac{2}{2}$, $\frac{1}{2}$ is the same as $\frac{2}{4}$, and so on), we've skipped over any redundant terms while weaving through the table.

The above process has allowed us to express the elements of $\mathbb{Q}_+$ in a list:

$$\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{3}{1}, \frac{1}{3}, \frac{1}{4}, \frac{2}{3}, \ldots$$

Similarly, we may list the elements of $\mathbb{Q}_-$—the set of *negative* rational numbers—by following an analogous process. By threading these two lists together like we did with the positive and negative integers in Example 2 (and putting 0 at the start), we obtain a single list for the elements of $\mathbb{Q}$:

$$0, \frac{1}{1}, -\frac{1}{1}, \frac{1}{2}, -\frac{1}{2}, \frac{2}{1}, -\frac{2}{1}, \frac{3}{1}, -\frac{3}{1}, \frac{1}{3}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{4}, \frac{2}{3}, -\frac{2}{3}, \ldots$$

Our list provides a one-to-one correspondence between $\mathbb{N}$ and $\mathbb{Q}$. Consequently, $\mathbb{Q}$ is countably infinite, as claimed. $\square$

We've just seen that the set of all rational numbers is countable, which should hopefully be a surprising fact. This begs the question: are there infinite sets that are *not* countable? That is, do there exist sets that are so large that their elements cannot be written down as a list? Remarkable, such sets (known as **uncountable** sets) do exist. Here is an example.

**Example 4:**

The interval $(0, 1)$ is an uncountable set.

*Proof:* What would it mean if $(0, 1)$ were, in fact, a *countable* set? Let's suppose for a moment that this were the case.

If $(0, 1)$ were countable, we could write the elements of this set as an infinite list. By expressing each number in $(0, 1)$ as a decimal, we obtain a list of the following form.

$$\begin{array}{c|l} 1 & 0\,.\,a_{11}\,a_{12}\,a_{13}\,a_{14}\,\dots \\ 2 & 0\,.\,a_{21}\,a_{22}\,a_{23}\,a_{24}\,\dots \\ 3 & 0\,.\,a_{31}\,a_{32}\,a_{33}\,a_{34}\,\dots \\ 4 & 0\,.\,a_{41}\,a_{42}\,a_{43}\,a_{44}\,\dots \\ \vdots & \qquad\quad\vdots \end{array}$$

Here, each decimal $a_{ij}$ a number in $\{0, 1, 2, \dots, 9\}$.

I claim that such a list can't possibly include *all* numbers from the interval $(0, 1)$, and I will show that this is the case by producing a number $b = 0.b_1\,b_2\,b_3\,b_4\,b_5\,\dots$ that is not on the list. We'll construct this number $b$ one decimal place at a time.

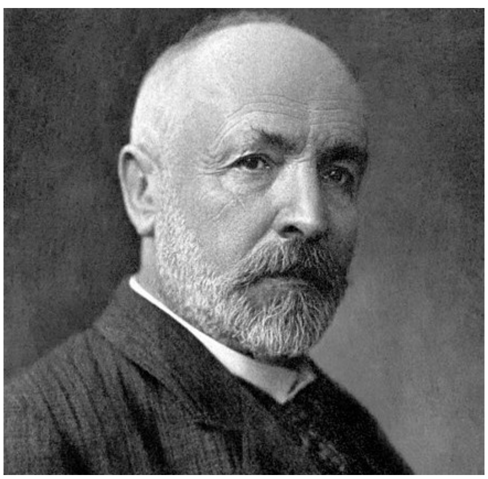
To ensure that $b$ is different from the first number on our list, we'll pick the first decimal place, $b_1$, as follows: if $a_{11} \neq 3$, choose $b_1 = 3$; and if $a_{11} = 3$, choose $b_1 = 4$. (Note that the choice of 3 and 4 is arbitrary here.)

Likewise, to ensure that $b$ is different from the second number on our list, we'll choose the second decimal place, $b_2$, as follows: if $a_{22} \neq 3$, choose $b_2 = 3$; and if $a_{22} = 3$, choose $b_2 = 4$.

Continue in this way, setting $b_i = 3$ if $a_{ii} \neq 3$ and $b_i = 4$ if $a_{ii} = 3$. This process ensures that the newly constructed number $b = 0.b_1\, b_2\, b_3\, b_4\, b_5\, \ldots$ is different from the $i^{th}$ number on our list in the $i^{th}$ decimal place. Thus, $b$ is nowhere on our list of numbers!

We have just shown that no list of numbers from the interval $(0, 1)$ will ever be complete. That is, it is impossible to establish a one-to-one correspondence between $(0, 1)$ and $\mathbb{N}$, hence $(0, 1)$ is an uncountable set. $\square$

A version of the above proof was published by mathematician Georg Cantor in 1891, providing one of the earliest examples of an uncountable set. This proof is known as **Cantor's diagonal argument**, as the number $b$ that is constructed differs from each element in the imagined list of numbers in the decimal place along the diagonal.



Georg Cantor: 1845-1918

$$\begin{array}{c|ccccccc}
1 & 0\,. & a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & \cdots \\
2 & 0\,. & a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & \cdots \\
3 & 0\,. & a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & \cdots \\
4 & 0\,. & a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & \cdots \\
5 & 0\,. & a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & \cdots \\
\vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

Cantor's diagonal argument

Cantor's results are counter-intuitive and, at the time, were considered to be highly controversial. He received harsh criticism from the mathematical and philosophical communities for suggesting that different sizes of infinity could exist. In time, however, Cantor's ideas became more widely accepted and are now staples of modern set theory. We will learn more about Cantor's theory of infinite sets in our next lesson.