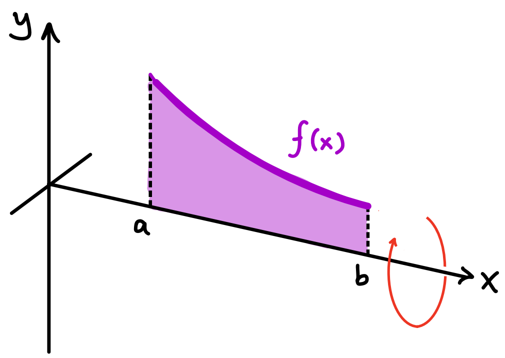
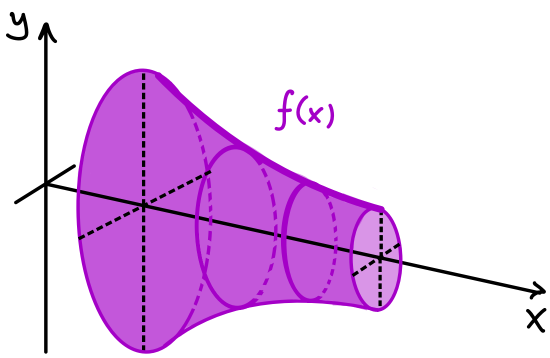
§3.2 – Volumes by Disks / Washers

Integrals don't just tell us about areas, we can also use them for calculating volumes of 3D Solids! The solids we'll consider in MATH 138 are called solids of revolution, and are obtained by revolving a 2D region about an axis:

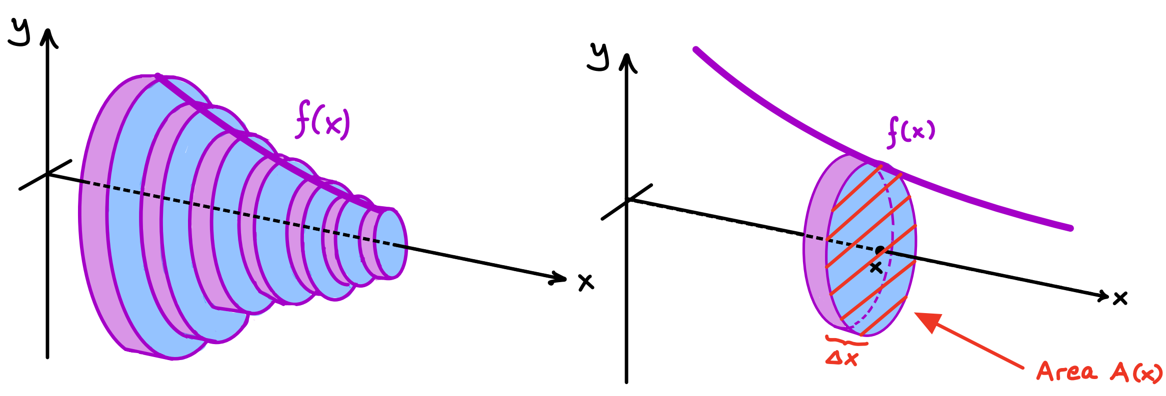




There are two ways to find the volume of such a solid.

1. The Disk / Washer Method

Start by slicing the solid into thin disks.



Each disk has width Δx. If $A(x)$ denotes the area of the disk at each point x, then the volume of a typical disk is $A(x)\Delta x$. Adding these volumes:

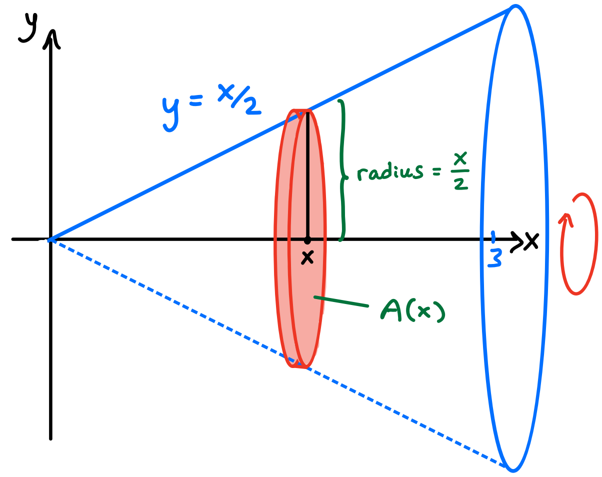
$$\boxed{\text{Volume of the Solid} = \int_a^b A(x)\,dx}$$

Ex: Consider the region between the x-axis and $y = \frac{x}{2}$ from $x = 0$ to $x = 3$. Find the volume of the solid obtained by rotating this region about the x-axis.

Solution: Start with a sketch showing the region and a typical disk. The area of a disk is

$A(x) = \pi \cdot \text{radius}^2 = \pi \cdot \left(\frac{x}{2}\right)^2,$

hence



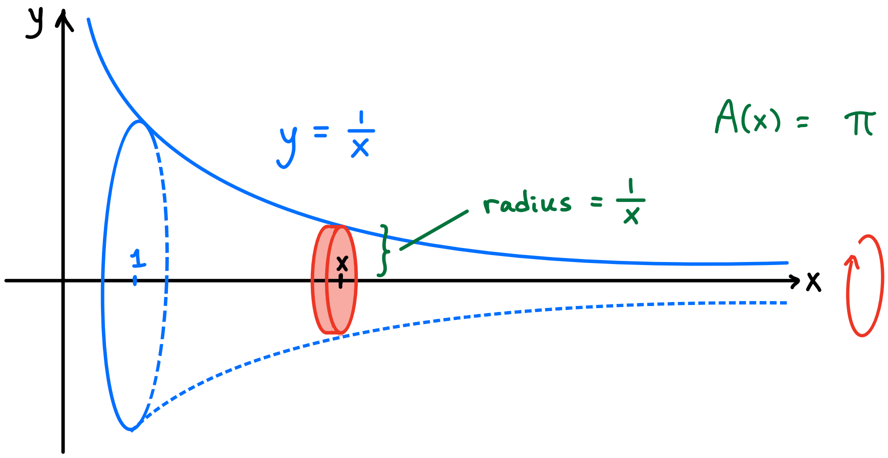
$$\text{Volume} = \int_0^3 A(x)\,dx = \int_0^3 \pi \left(\frac{x}{2}\right)^2 dx$$

$$= \frac{\pi}{4}\int_0^3 x^2\,dx$$

$$= \frac{\pi}{4}\left[\frac{x^3}{3}\right]_0^3 = \boxed{\frac{9\pi}{4}}$$

Example [Gabriel's Horn]:

Let R denote the region between $y = \frac{1}{x}$ and the x-axis for $x \in [1, \infty)$. Find the volume of the solid obtained by rotating R about the x-axis. This solid is known as Gabriel's Horn.



$$A(x) = \pi\,(\text{radius})^2 = \frac{\pi}{x^2}$$

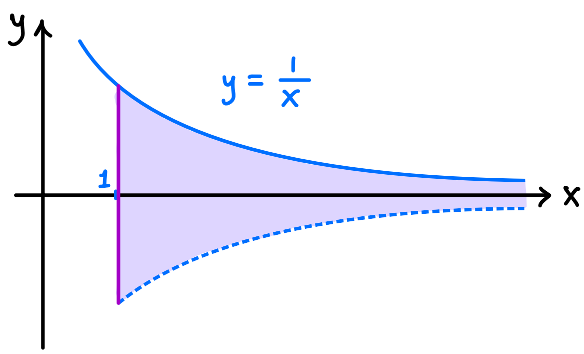
$$\text{Volume} = \int_1^\infty A(x)\,dx = \int_1^\infty \frac{\pi}{x^2}\,dx$$

$$= \lim_{t\to\infty} \left[-\frac{\pi}{x^2}\right]_1^t$$

$$= \lim_{t\to\infty} \left[-\frac{\pi}{t^2} + \frac{\pi}{1^2}\right] = \boxed{\pi}$$

[Note: The (lengthwise) cross-sectional area of the horn is

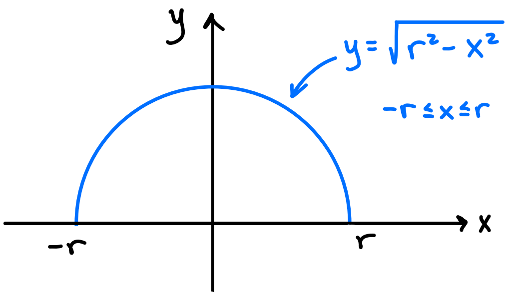
$$2\int_1^\infty \frac{1}{x}\,dx = \infty \quad \text{(divergent).}$$



It may therefore be surprising that the horn encloses a finite volume!]

Ex: Find the volume of a sphere of radius r.

Solution: A sphere can be obtained by rotating the top half of the circle $x^2+y^2=r^2$ about the x-axis!



$$\text{Volume} = \int_{-r}^{r} A(x)\,dx$$

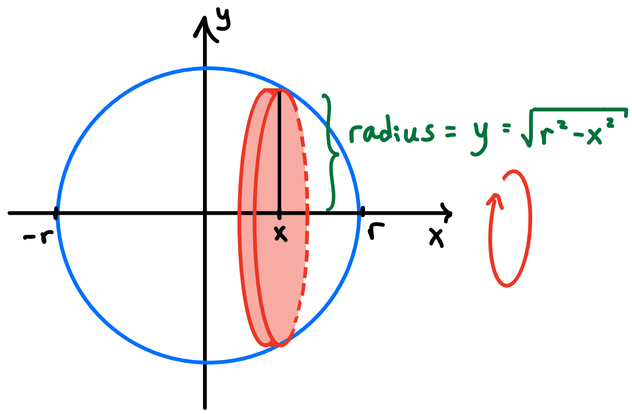
$$= \int_{-r}^{r} \pi \cdot \text{radius}^2\,dx$$

$$= \int_{-r}^{r} \pi\left(\sqrt{r^2-x^2}\right)^2 dx$$

$$= \int_{-r}^{r} \pi\left(r^2-x^2\right) dx$$

$$= \left[\pi r^2 x - \frac{\pi x^3}{3}\right]_{-r}^{r}$$

$$= \left(\pi r^3 - \frac{\pi r^3}{3}\right) - \left(-\pi r^3 + \frac{\pi r^3}{3}\right) = \boxed{\frac{4\pi r^3}{3}}$$

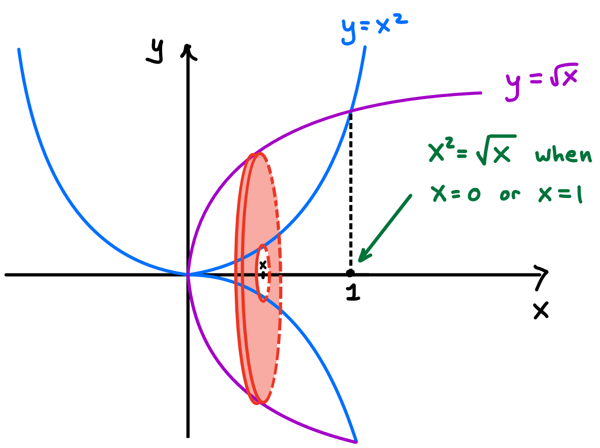


Ex: Set up the integral that gives the volume of the solid obtained by rotating each region about the given axis.

(a) Region: bounded between $y=x^2$ and $y=\sqrt{x}$

Axis: x-axis.

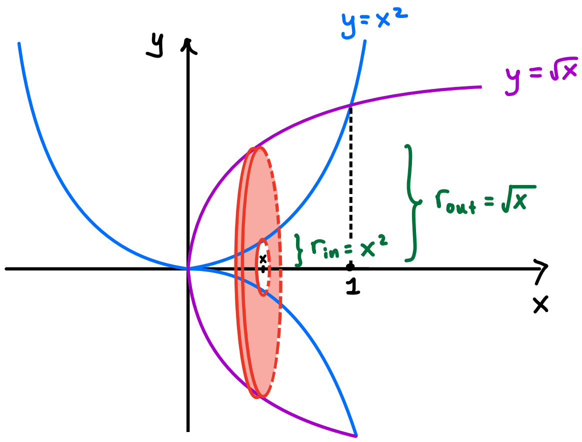
Solution: Start with a sketch!



This time our cross-section isn't a disk... it's a washer!

In this case,

$$\boxed{\text{Area} = A(x) = \pi \cdot (\text{outer radius})^2 - \pi(\text{inner radius})^2}$$



Outer radius $=\sqrt{x}$

Inner radius $=x^2$

<u>Bounds:</u> $0 \leq x \leq 1$

$$\therefore \text{Volume} = \int_0^1 A(x)\,dx = \int_0^1 \left[\pi(r_{out})^2 - \pi(r_{in})^2\right]dx$$

$$= \boxed{\int_0^1 \left[\pi(\sqrt{x})^2 - \pi(x^2)^2\right]dx}$$

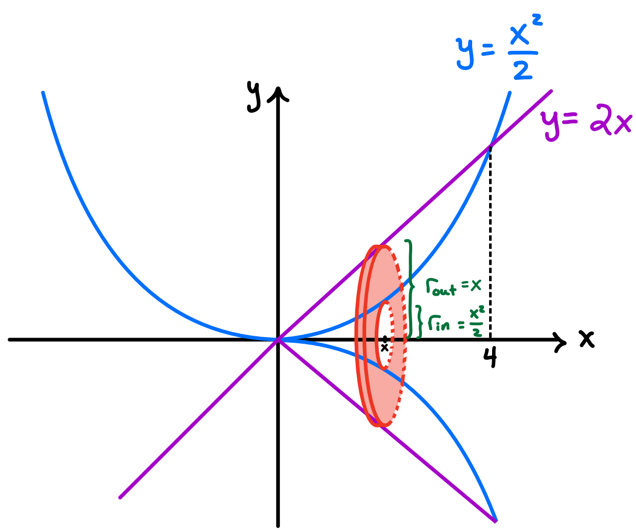
(b) <u>Region:</u> bounded between $y=2x$ and $y=\frac{x^2}{2}$

<u>Axis:</u> x-axis.

<u>Solution:</u> Sketch!

Outer radius: $2x$

Inner radius: $\frac{x^2}{2}$



Bounds?

$$\frac{x^2}{2} = 2x \Rightarrow x^2 = 4x \Rightarrow x(x-4) = 0 \Rightarrow x = 0 \text{ or } x = 4.$$

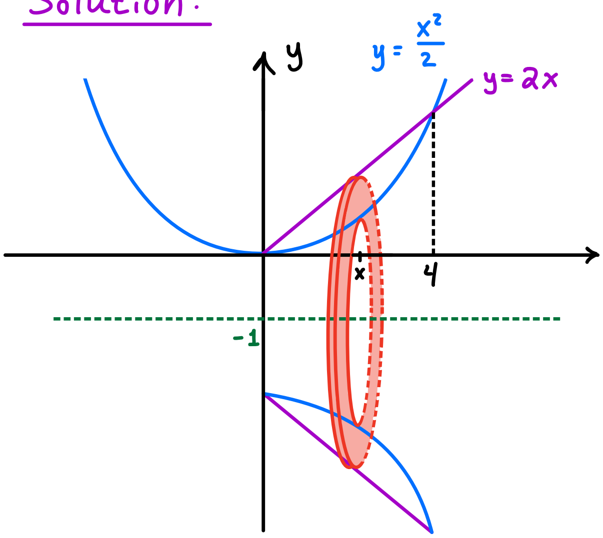
$$\therefore \text{ Volume } = \int_0^4 A(x)\,dx = \int_0^4 \left[\pi (r_{out})^2 - \pi (r_{in})^2\right] dx$$

$$= \boxed{\int_0^4 \left[\pi (2x)^2 - \pi\left(\frac{x^2}{2}\right)^2\right] dx}$$

(c) Region: bounded between $y = 2x$ and $y = \frac{x^2}{2}$

Axis: $y = -1$

Solution:



Outer radius $= 1 + 2x$

Inner radius $= 1 + \frac{x^2}{2}$

Bounds: $0 \le x \le 4$

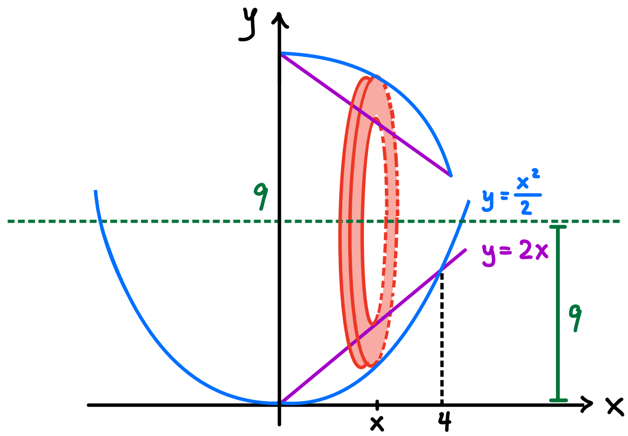
(same as before)

$$\therefore \text{ Volume } = \int_0^4 A(x)\,dx = \boxed{\int_0^4 \left[\pi (1+2x)^2 - \pi\left(1 + \frac{x^2}{2}\right)^2\right] dx}$$

(d) Region: bounded between $y=2x$ and $y=\frac{x^2}{2}$

Axis: $y=9$

Solution:



Outer radius: $9-\frac{x^2}{2}$

Inner radius: $9-2x$

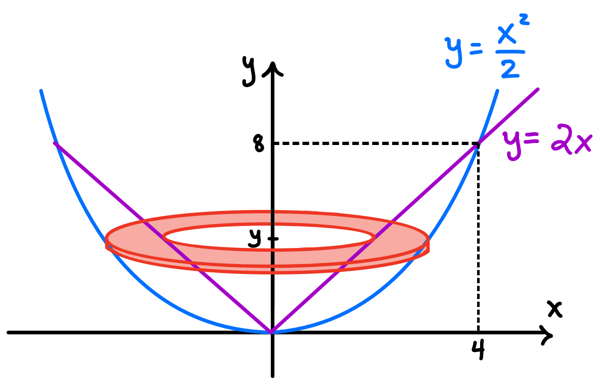
Bounds: $0 \le x \le 4$.

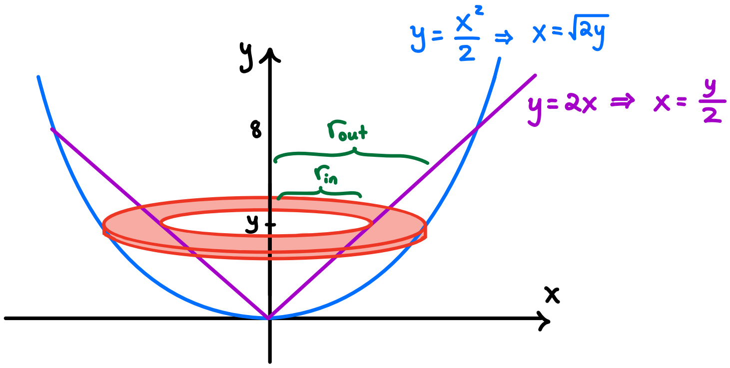
$$\therefore \text{ Volume } = \int_0^4 A(x)\,dx = \boxed{\int_0^4 \left[\pi\left(9-\frac{x^2}{2}\right)^2 - \pi(9-2x)^2\right]dx}$$

(e) Region: bounded between $y=2x$ and $y=\frac{x^2}{2}$

Axis: y-axis

Solution: We have a washer at each $y \in [0,8]$, so we'll be integrating with respect to y.





Outer radius: $\sqrt{2y}$

Inner radius: $\frac{y}{2}$

Bounds: $0 \le y \le 8$.

$$\therefore \text{Volume} = \int_0^8 A(y)\,dy = \boxed{\int_0^8 \left[\pi\left(\sqrt{2y}\right)^2 - \pi\left(\frac{y}{2}\right)^2\right] dy}$$

Summary for disks / washers

Revolving around horizontal axis? Use functions of x.

Revolving around vertical axis? Use functions of y.