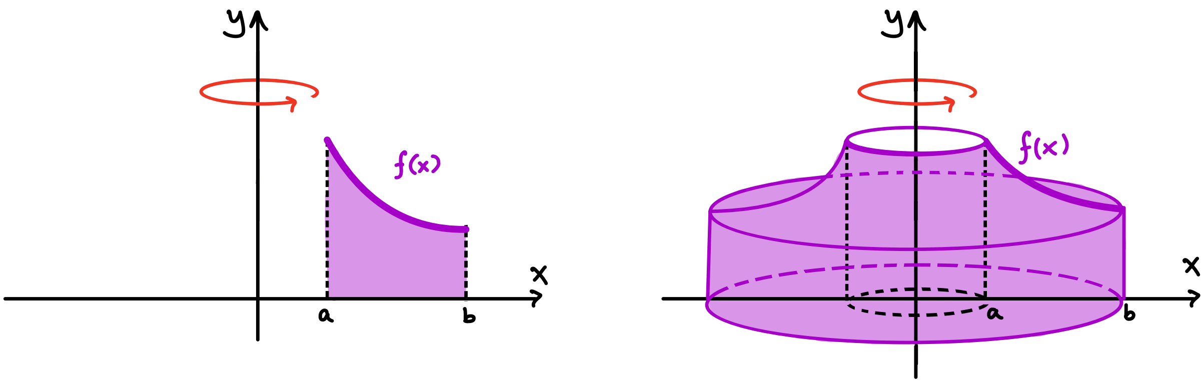
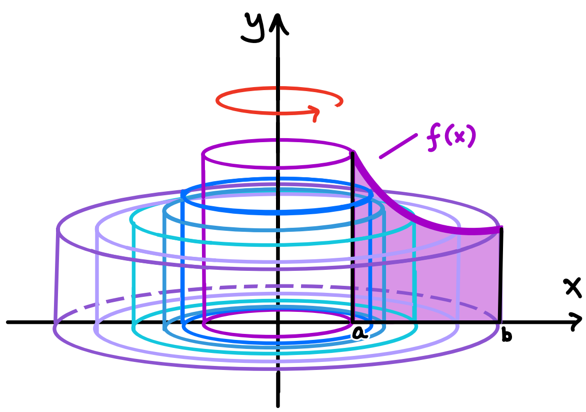
§ 3.3 – Volumes by Cylindrical Shells

Suppose we wish to compute the volume of the solid of revolution below.



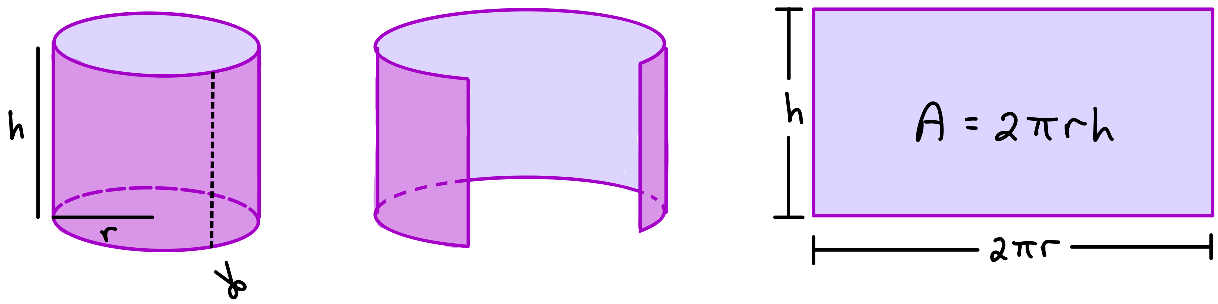
This time, we'll split up the solid into very thin cylindrical shells.



We again have

$$\boxed{\text{Volume} = \int_a^b A(x)\,dx}$$

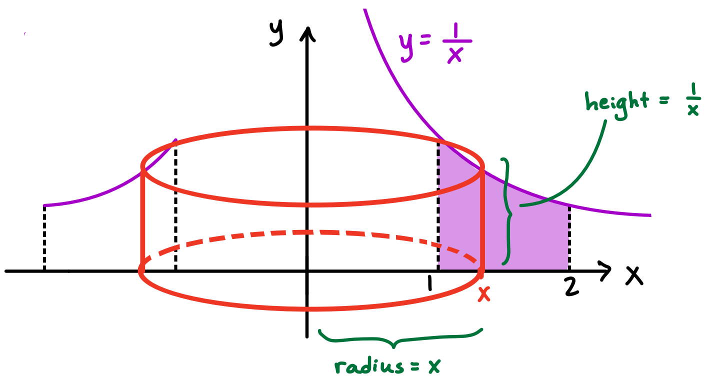
but now $A(x)$ is the surface area of a typical cylindrical shell.



$$\boxed{A(x) = 2\pi \cdot r \cdot h}$$

Ex: Let R denote the region between $y = 0$ and $y = \frac{1}{x}$ from $x = 1$ to $x = 2$. Find the volume of the solid obtained by rotating R about the y-axis.

Solution:



$$\text{Volume} = \int_1^2 A(x)\,dx = \int_1^2 2\pi r h\,dx$$

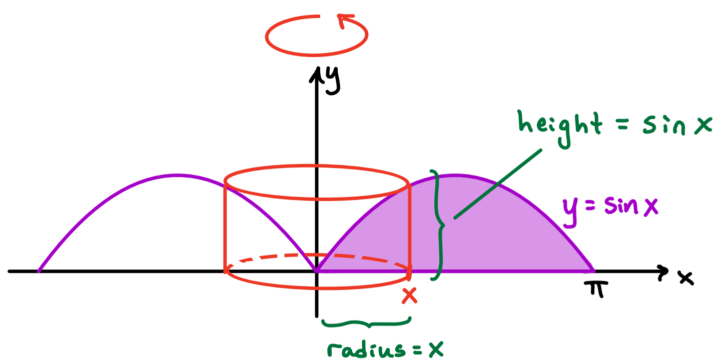
$$= \int_1^2 2\pi \cdot x \cdot \frac{1}{x}\,dx = \int_1^2 2\pi\,dx = \boxed{2\pi}$$

Ex: Set up the integral that gives the volume of the solid obtained by rotating each region about the given axis.

(a) Region: Between $y = \sin x$ and $y = 0$, $0 \le x \le \pi$

Axis: y-axis.

Solution:

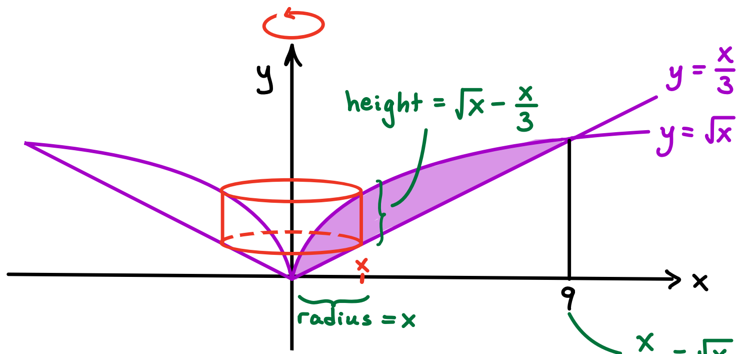


$$\text{Volume} = \int_0^\pi A(x)\,dx = \int_0^\pi 2\pi r h\,dx = \boxed{\int_0^\pi 2\pi x \cdot \sin x\,dx}$$

(b) Region: Bounded between $y=\sqrt{x}$ and $y=\frac{x}{3}$.

Axis: y-axis.

Solution:

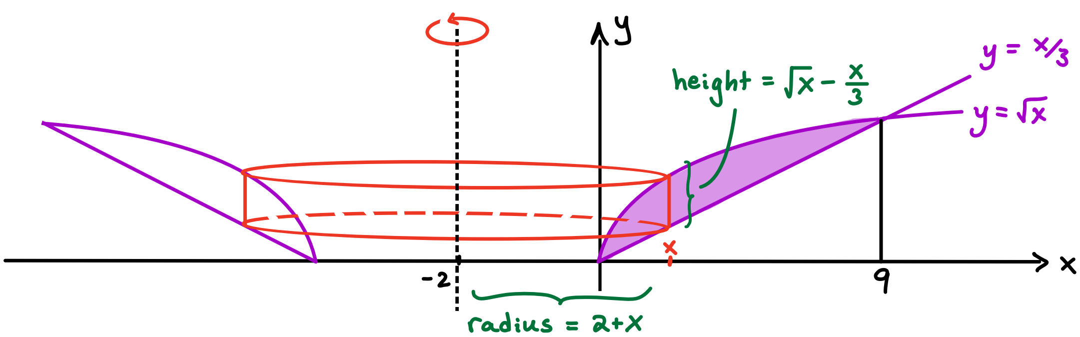


$$V = \int_0^9 A(x)\,dx = \int_0^9 2\pi r h\,dx = \boxed{\int_0^9 2\pi x\left(\sqrt{x}-\frac{x}{3}\right)dx}$$

(c) Region: Bounded between $y=\sqrt{x}$ and $y=\frac{x}{3}$.

Axis: $x=-2$

Solution:

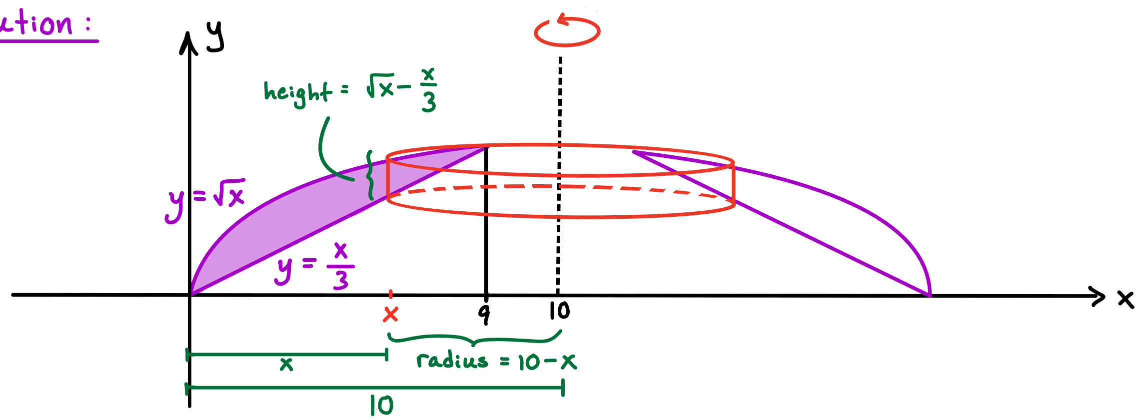


$$V = \int_0^9 A(x)\,dx = \int_0^9 2\pi r h\,dx = \boxed{\int_0^9 2\pi\,(2+x)\left(\sqrt{x}-\frac{x}{3}\right)dx}$$

(d) Region: Bounded between $y=\sqrt{x}$ and $y=\dfrac{x}{3}$.

Axis: $x = 10$

Solution:

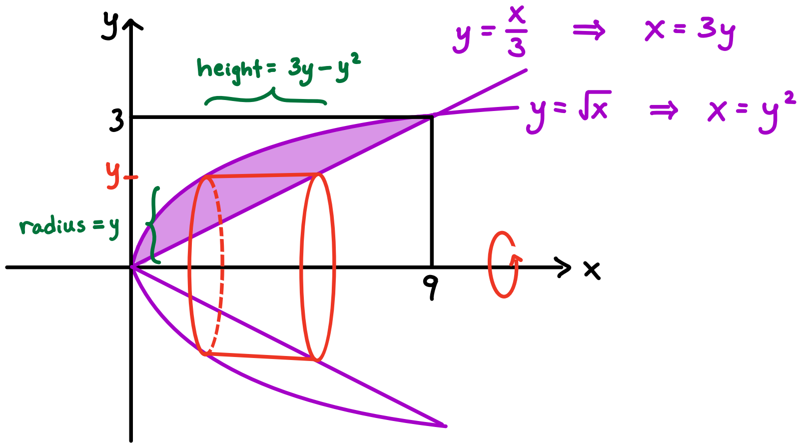


$$V = \int_0^9 A(x)\,dx = \int_0^9 2\pi r h\,dx = \boxed{\int_0^9 2\pi\,(10-x)\left(\sqrt{x}-\frac{x}{3}\right)dx}$$

(e) Region: Bounded between $y=\sqrt{x}$ and $y=\dfrac{x}{3}$.

Axis: x-axis (but do it with shells!)

Solution: We get a cylindrical shell for each $y\in[0,3]$, so we'll be integrating with respect to y.



$$V = \int_0^3 A(y)\,dy = \int_0^3 2\pi r h\,dy = \boxed{\int_0^3 2\pi y(3y - y^2)\,dy}$$

Summary for cylindrical shells

Revolving around vertical axis? Use functions of x.

Revolving around horizontal axis? Use functions of y.