§2.1 – Trigonometric Substitution

Some integrals can be simplified if we think of x

as a trigonometric function!

Intro Example:

$$\int \frac{1}{\sqrt{1-x^2}}\,dx$$

From our library of antiderivatives, we know this is $\arcsin(x) + C$... but let's see a different way!

Let $x = \sin\theta$, so $dx = \cos\theta\, d\theta$. We have

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \int \frac{1}{\sqrt{1-\sin^2\theta}} \cdot \cos\theta\, d\theta$$

$$= \int \frac{\cos\theta}{\sqrt{\cos^2\theta}}\, d\theta$$

$$= \int \frac{\cos\theta}{\cos\theta}\, d\theta = \int 1\, d\theta = \theta + C$$

Note: $x = \sin\theta \implies \theta = \arcsin(x)$ $\left(\text{for } -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\right)$

$$\therefore \int \frac{1}{\sqrt{1-x^2}}\,dx = \theta + C = \boxed{\arcsin(x) + C}$$

The process above is called a trigonometric substitution!

u-substitution: $u = g(x)$, $du = g'(x)\,dx$

Trig substitution: $x = g(\theta)$, $dx = g'(\theta)\,d\theta$

If you see ...	try substituting ...	domain for θ
$\sqrt{a^2 - x^2}$ ($a \in \mathbb{R}$)	$x = a \cdot \sin\theta$	$-\pi/2 \le \theta \le \pi/2$
$\sqrt{a^2 + x^2}$	$x = a \cdot \tan\theta$	$-\pi/2 < \theta < \pi/2$
$\sqrt{x^2 - a^2}$	$x = a \cdot \sec\theta$	$0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$

The range for θ guarantees that the trig function can be inverted to express θ in terms of x. You should know these ranges, but you don't need to write them.

Ex: Evaluate the following.

(a) $\displaystyle\int \frac{1}{\sqrt{x^2+9}}\,dx$

Solution: Let $x = 3\tan\theta$, so $dx = 3\sec^2\theta\,d\theta$. Thus,

$$\int \frac{1}{\sqrt{x^2+9}}\,dx = \int \frac{1}{\sqrt{9\tan^2\theta+9}} \cdot 3\sec^2\theta\,d\theta$$

$$= \int \frac{3\sec^2\theta}{\sqrt{9(\tan^2\theta+1)}}\,d\theta$$

$$= \int \frac{\cancel{3}\sec^2\theta}{\cancel{3}\sqrt{\sec^2\theta}}\,d\theta$$

$$= \int \frac{\sec^2\theta}{|\sec\theta|}\,d\theta$$

$$= \int \frac{\sec^2\theta}{\sec\theta}\,d\theta$$

Our range for θ is $-\pi/2 < \theta < \pi/2$, and $\sec\theta > 0$ on this range.

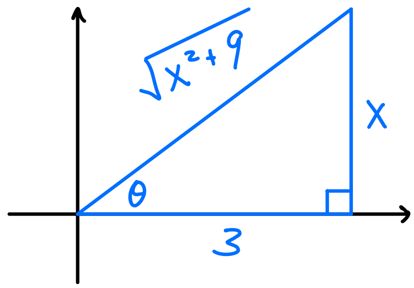
$\therefore\ |\sec\theta| = \sec\theta$.

(absolute value will always disappear in a trig sub!)

$$= \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C.$$

[To go back to x's, draw a triangle for $x = 3\tan\theta$.]

$$x = 3\tan\theta \Rightarrow \tan\theta = \frac{x}{3} = \frac{\text{opposite}}{\text{adjacent}}$$



Hence, $\sec\theta = \dfrac{\text{hypotenuse}}{\text{adjacent}} = \dfrac{\sqrt{x^2+9}}{3}$

Thus, $$\int \frac{1}{\sqrt{x^2+9}}\,dx = \ln\left|\sec\theta + \tan\theta\right| + C$$

$$= \boxed{\ln\left|\frac{\sqrt{x^2+9}}{3} + \frac{x}{3}\right| + C}$$

(b) $$\int \frac{dx}{x^2\sqrt{x^2-2}}$$

<u>Solution:</u> Let $x = \sqrt{2}\sec\theta$, $dx = \sqrt{2}\sec\theta\tan\theta\,d\theta$. Thus,

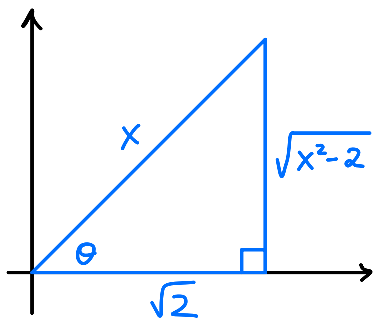
$$\int \frac{dx}{x^2\sqrt{x^2-2}} = \int \frac{\sqrt{2}\sec\theta\tan\theta}{2\sec^2\theta\cdot\sqrt{2\sec^2\theta - 2}}\,d\theta$$

$$= \int \frac{\tan\theta}{\sqrt{2}\sec\theta\sqrt{2(\sec^2\theta - 1)}}\,d\theta$$

$$= \frac{1}{\sqrt{2}} \int \frac{\cancel{\tan\theta}}{\sec\theta \cdot \sqrt{2}\,\cancel{\tan\theta}}\, d\theta$$

$$= \frac{1}{2} \int \frac{1}{\sec\theta}\, d\theta$$

$$= \frac{1}{2} \int \cos\theta \, d\theta = \frac{\sin\theta}{2} + C$$



Now, going back to x's

$$x = \sqrt{2}\sec\theta \Rightarrow \sec\theta = \frac{x}{\sqrt{2}} = \frac{\text{hypotenuse}}{\text{adjacent}}$$

Hence, $\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{\sqrt{x^2-2}}{x}$.

$$\therefore \int \frac{1}{x^2\sqrt{x^2-2}}\, dx = \frac{\sin\theta}{2} + C = \boxed{\frac{\sqrt{x^2-2}}{2x} + C}$$

(c) $\displaystyle\int \frac{\sqrt{1-4x^2}}{x^2}\, dx$

Solution: We'll first manipulate the integrand ...

$$\int \frac{\sqrt{1-4x^2}}{x^2}\, dx = \int \frac{\sqrt{4\left(\frac{1}{4}-x^2\right)}}{x^2}\, dx$$

⟵ let $x = \frac{1}{2}\sin\theta$

$dx = \frac{1}{2}\cos\theta\, d\theta$

$$= \int \frac{\sqrt{4\left(\frac{1}{4} - \frac{1}{4}\sin^2\theta\right)}}{\frac{1}{4}\sin^2\theta} \cdot \frac{1}{2}\cos\theta\, d\theta$$

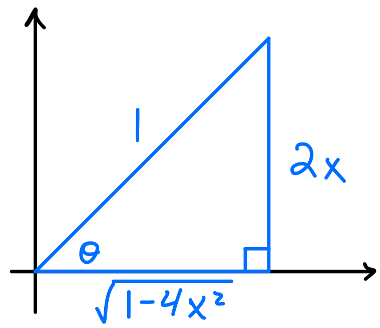
$$= \int \frac{\sqrt{\cancel{4}\cdot\cancel{\frac{1}{4}}\cos^2\theta}}{\frac{1}{4}\sin^2\theta} \cdot \frac{1}{2}\cos\theta\, d\theta$$

$$= \int \frac{1/2\,\cos^2\theta}{1/4\,\sin^2\theta}$$

$$= 2\int \cot^2\theta\, d\theta$$

$$= 2\int \left(\csc^2\theta - 1\right) d\theta = -2\cot\theta - 2\theta + C$$

<u>Now, going back to x's ...</u>



$$x = \frac{1}{2}\sin\theta \Rightarrow \sin\theta = \frac{2x}{1} = \frac{\text{opposite}}{\text{hypotenuse}}$$

Hence, $\cot\theta = \dfrac{\text{adjacent}}{\text{opposite}} = \dfrac{\sqrt{1-4x^2}}{2x}$ and $\theta = \arcsin(2x)$

$$\therefore \int \frac{\sqrt{1-4x^2}}{x^2}\, dx = -2\cot\theta - 2\theta + C$$

$$= \boxed{\frac{-\sqrt{1-4x^2}}{x} - 2\arcsin(2x) + C}$$

Note: Trig subs can work even without square roots!

(d) $\displaystyle\int_0^1 \frac{x^2}{(1+x^2)^2}\,dx$

Solution: Let $x = \tan\theta$, so $dx = \sec^2\theta\,d\theta$

When $x = \tan\theta = 0$, we have $\theta = 0$

When $x = \tan\theta = 1$, we have $\theta = \pi/4$

Remember: $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

Thus,

$$\int_0^1 \frac{x^2}{(1+x^2)^2}\,dx = \int_0^{\pi/4} \frac{\tan^2\theta}{(1+\tan^2\theta)^2}\sec^2\theta\,d\theta$$

$$= \int_0^{\pi/4} \frac{\tan^2\theta}{\sec^4\theta}\sec^2\theta\,d\theta$$

$$= \int_0^{\pi/4} \frac{\tan^2\theta}{\sec^2\theta}\,d\theta$$

$$= \int_0^{\pi/4} \frac{\left(\sin^2\theta/\cos^2\theta\right)}{\left(1/\cos^2\theta\right)}\,d\theta$$

$$= \int_0^{\pi/4} \sin^2\theta\,d\theta$$

Helpful Identities

$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

Note:

$\int \cos(n\theta)\,d\theta = \frac{\sin n\theta}{n} + C$

(Prove this!)

$$= \int_0^{\pi/4} \frac{1}{2}(1-\cos 2\theta)\,d\theta$$

$$= \frac{1}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/4}$$

$$= \frac{1}{2}\left[\frac{\pi}{4} - \frac{\sin \pi/2}{2}\right] - \frac{1}{2}\left[0 - \underbrace{\frac{\sin 0}{2}}_{=0}\right]$$

$$= \boxed{\frac{\pi}{8} - \frac{1}{4}}$$

You may need to complete the square before making a trig sub.

e.g. $x^2 - 8x + 2 = (x^2 \underbrace{-8x + 16}_{\div 2 \text{ and square}}) + 2 - 16 = (x-4)^2 - 14$

e.g. $2x^2 + 12x - 7 = 2(x^2 + \underbrace{6x + 9}_{\div 2 \text{ and square}}) - 7 - 18 = 2(x+1)^2 - 25$

Ex: $\displaystyle\int \frac{1}{(x^2+2x+5)^{3/2}}\,dx$

Solution: Completing the square, we have

$$\int \frac{1}{(x^2+2x+5)^{3/2}}\,dx = \int \frac{1}{\left((x+1)^2+4\right)^{3/2}}\,dx$$

Let $x+1 = 2\tan\theta$

$dx = 2\sec^2\theta\, d\theta$

$$= \int \frac{1}{(4\tan^2\theta+4)^{3/2}} \cdot 2\sec^2\theta\, d\theta$$

$$= \int \frac{2\sec^2\theta}{4^{3/2}(\sec^2\theta)^{3/2}}\, d\theta$$

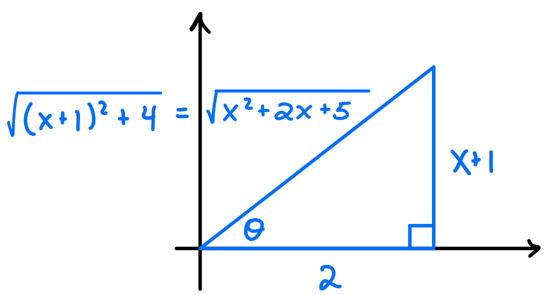
$$= \int \frac{2\sec^2\theta}{8\sec^3\theta}\, d\theta$$

$$= \frac{1}{4}\int \cos\theta\, d\theta = \frac{\sin\theta}{4} + C.$$

Now, going back to x's ...

$$x+1 = 2\tan\theta \implies \tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{x+1}{2}.$$

We have $\sin\theta = \dfrac{x+1}{\sqrt{x^2+2x+5}}$



$$\Rightarrow \int \frac{1}{(x^2+2x+5)^{3/2}}\,dx = \frac{\sin\theta}{4} + C = \boxed{\frac{x+1}{4\sqrt{x^2+2x+5}} + C}$$

$\underline{Ex}$: $\displaystyle\int \frac{x}{\sqrt{x^2+2x-8}}\,dx$

Solution: Completing the square, we have

$$\int \frac{x}{\sqrt{x^2+2x-8}}\,dx = \int \frac{x}{\sqrt{(x+1)^2-9}}\,dx$$

Let $x+1 = 3\sec\theta$
$(x = 3\sec\theta - 1)$

then $dx = 3\sec\theta\tan\theta\,d\theta$

$$= \int \frac{3\sec\theta - 1}{\sqrt{9\sec^2\theta - 9}} \cdot 3\sec\theta\tan\theta\,d\theta$$

$$= \int \frac{3\sec\theta - 1}{\cancel{\sqrt{9}}\sqrt{\sec^2\theta - 1}} \cdot \cancel{3}\sec\theta\tan\theta\,d\theta$$

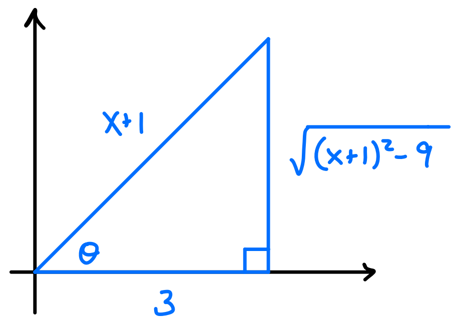
$$= \int \frac{3\sec\theta - 1}{\cancel{\sqrt{\tan^2\theta}}} \cdot \sec\theta\,\cancel{\tan\theta}\,d\theta$$

$$= \int (3\sec^2\theta - \sec\theta)\,d\theta$$

$$= 3\tan\theta - \ln|\sec\theta + \tan\theta| + C$$

Now, going back to x's ...

$$x+1 = 3\sec\theta \implies \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{x+1}{3}.$$



Hence,

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sqrt{(x+1)^2-9}}{3} = \frac{\sqrt{x^2+2x-8}}{3},$$

and therefore

$$\int \frac{x}{\sqrt{x^2+2x-8}}\,dx = 3\tan\theta - \ln\left|\sec\theta+\tan\theta\right| + C$$

$$= \boxed{\sqrt{x^2+2x-8} - \ln\left|\frac{x+1}{3} + \frac{\sqrt{x^2+2x-8}}{3}\right| + C}$$

<u>Additional Exercise:</u> $\displaystyle\int \frac{x}{(5-4x-x^2)^{3/2}}\,dx$

<u>Solution:</u> Complete the square to write

$$5-4x-x^2 = -(x^2+4x+4)+5+4 = 9-(x+2)^2$$

Hence,

$$\int \frac{x}{(5-4x-x^2)^{3/2}}\,dx = \int \frac{x}{\left(9-(x+2)^2\right)^{3/2}}\,dx$$

Let $x+2 = 3\sin\theta$

$(x = 3\sin\theta - 2)$

then $dx = 3\cos\theta\, d\theta$

$$= \int \frac{3\sin\theta - 2}{(9-9\sin^2\theta)^{3/2}} \cdot 3\cos\theta\, d\theta$$

$$= \int \frac{3\sin\theta - 2}{9^{3/2}(1-\sin^2\theta)^{3/2}} \cdot 3\cos\theta\, d\theta$$

$$= \int \frac{3\sin\theta - 2}{27(\cos^2\theta)^{3/2}} \cdot 3\cos\theta\, d\theta$$

$$= \frac{1}{9}\int \frac{3\sin\theta - 2}{\cos^3\theta} \cos\theta\, d\theta$$

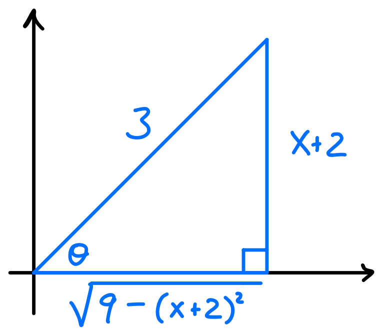
$$= \frac{1}{9}\int \left(\frac{3}{\cos\theta}\cdot\frac{\sin\theta}{\cos\theta} - \frac{2}{\cos^2\theta}\right) d\theta$$

$$= \frac{1}{9}\int \left(3\sec\theta\tan\theta - 2\sec^2\theta\right) d\theta$$

$$= \frac{\sec\theta}{3} - \frac{2}{9}\tan\theta + C$$

Now, going back to x's ...

$$x+2 = 3\sin\theta \quad \Rightarrow \quad \sin\theta = \frac{x+2}{3} = \frac{\text{opposite}}{\text{hypotenuse}}$$



Hence, $\sec\theta = \dfrac{\text{hypotenuse}}{\text{adjacent}} = \dfrac{3}{\sqrt{9-(x+2)^2}} = \dfrac{3}{\sqrt{5-4x-x^2}}$

and $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}} = \dfrac{x+2}{\sqrt{9-(x+2)^2}} = \dfrac{x+2}{\sqrt{5-4x-x^2}}$

Thus, $$\int \frac{x}{(5-4x-x^2)^{3/2}}\,dx = \frac{\sec\theta}{3} - \frac{2}{9}\tan\theta + C$$

$$= \boxed{\frac{1}{\sqrt{5-4x-x^2}} - \frac{2(x+2)}{9\sqrt{5-4x-x^2}} + C}$$