§ 6.7, 6.8 – Taylor and Maclaurin Series

Previously we learned a variety of tricks for manipulating the power series representation for a function f (up to now, either $\frac{1}{1-x}$ or e^x) to obtain a power series representation for a related function.

In this section, we'll learn more about which functions actually have representations as power series and, if such a representation does exist, we'll determine a formula for its coefficients!

Suppose that f has a power series representation, centred at $x=a$, say

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots$$

for all x with $|x-a| < R$, $R>0$

Radius of convergence!

Note that at $x = a$:

$$f(a) = a_0 + 0 + 0 + \cdots \Rightarrow \boxed{a_0 = f(a)}$$

Differentiating f and its power series, we have

$$f'(x) = a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + \cdots$$

and so at $x = a$:

$$f'(a) = a_1 + 0 + 0 + \cdots \Rightarrow \boxed{a_1 = f'(a)}$$

Differentiating again, we have

$$f''(x) = a_2 + 3\cdot 2\cdot(x-a) + 4\cdot 3\cdot(x-a)^2 + \cdots$$

and hence at $x = a$:

$$f''(a) = 2a_2 + 0 + 0 + \cdots \Rightarrow \boxed{a_2 = \frac{f''(a)}{2}}$$

If we were to repeat this, we would find that

$$\boxed{a_3 = \frac{f'''(a)}{3!}}$$

In general, we obtain

$$\boxed{a_n = \frac{f^{(n)}(a)}{n!} \text{ for all } n.}$$

We have just proven the following theorem!

Theorem: If f has a power series representation centred at a, say $f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$ for $|x-a| < R$ where $R > 0$, then $a_n = \frac{f^{(n)}(a)}{n!}$.

That is,

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n}$$

Definition: The series $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ is called the Taylor series for f centred at $x=a$. In the special case that $a=0$, we call $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$ the Maclaurin series for f.

Remark: This theorem tells us that the coefficients in a power series representation for f aren't random — they encode information about the derivatives of f!

Ex: What is the Taylor series for $f(x) = e^{-x^2}$ centred at $x=0$ (i.e., the Maclaurin series for f)?

Solution: Rather than directly calculating $f^{(n)}(0)$, let's use manipulations!

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}}$$

By the previous theorem, this power series representation is the Maclaurin series for f!

Follow up: What is $f^{(20)}(0)$ for $f(x) = e^{-x^2}$?

Solution: In the Maclaurin series, the coefficient of

x^{20} is $\frac{f^{(20)}(0)}{20!}$. We showed that the Maclaurin series for $f(x) = e^{-x^2}$ is $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$, and we see that x^{20} is obtained when $2n = 20$ or $n = 10$. Hence, setting $n = 10$, the coefficient of x^{20} is

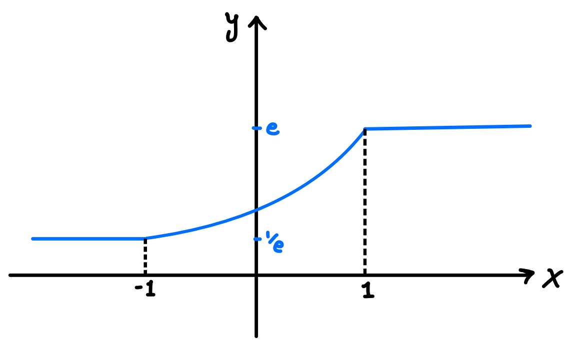
$$\frac{(-1)^{10}}{10!} = \frac{1}{10!} \Rightarrow \frac{1}{10!} = \frac{f^{(20)}(0)}{20!}$$

$$\Rightarrow \boxed{f^{(20)}(0) = \frac{20!}{10!}}$$

<u>Remark</u>: The previous theorem assumes that f can be represented as a power series, but is this always possible? No! Here's an example!

<u>Ex</u>: Consider the function

$$f(x) = \begin{cases} 1/e & \text{if } x < -1 \\ e^x & \text{if } -1 \le x \le 1 \\ e & \text{if } x > 1 \end{cases}$$



If this function <u>did</u> have a power series representation at 0, it would have to be its Maclaurin series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

But near $x=0$, we have $f^{(n)}(x) = e^x$ for all n, hence

the Maclaurin series for f is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{e^0}{n!} x^n$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x \text{ for all } x.$$

So, even though the Maclaurin series for f converges for all $x \in (-\infty, \infty)$, it only represents f for $x \in [-1,1]$.

<u>This begs the question:</u>

Which functions are equal to their Taylor series for all x in the interval of convergence?

Since the partial sums of the Taylor series

$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ are exactly the Taylor polynomials

$T_{n,a}(x)$, we are really asking:

For which functions f is it true that

$$f(x) = \lim_{n\to\infty} T_{n,a}(x) \ ?$$

Since $f(x) = T_{n,a}(x) + R_{n,a}(x)$, we are really asking:

The error / remainder

For which functions f is it true that

$$\lim_{n\to\infty} R_{n,a}(x) = 0 \ ?$$

The following theorem provides the answer!

<u>Theorem</u> [Convergence of Taylor Series]:

Suppose f has derivatives of all orders on an interval I

containing $x=a$. If there exists a constant $M \in \mathbb{R}$ with

$|f^{(n)}(x)| \leq M$ for all $n \in \mathbb{N}$ and all $x \in I$, then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \quad \text{for all } x \in I.$$

<u>Proof</u>: Let $x \in I$. Since $|f^{(n)}(x)| \leq M$ for all n, by Taylor's Inequality,

$$0 \leq |R_{n,a}(x)| \leq \frac{M|x-a|^{n+1}}{(n+1)!}.$$

As $n \to \infty$, we have $\lim_{n\to\infty} \frac{M|x-a|^{n+1}}{(n+1)!} = 0$.

[Why? Well... we have seen that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all x, hence $\sum_{n=0}^{\infty} \frac{(x-a)^n}{n!}$ converges for all x. Thus, by the Divergence Test, $\lim_{n\to\infty} \frac{(x-a)^n}{n!} = 0$.]

By the squeeze theorem, $\lim_{n\to\infty} R_{n,a}(x) = 0$, hence $f(x)$ is equal to its Taylor series on I, as desired. ■

Corollary: $\sin x$ and $\cos x$ are equal to their Taylor series for all $x \in (-\infty, \infty)$.

Proof: $\sin x$ and $\cos x$ are infinitely differentiable on $(-\infty, \infty)$ with all derivatives ($\pm \sin x$ and $\pm \cos x$) always bounded by $M = 1$. ∎

Ex: Find the Taylor series for $f(x) = \cos x$ centred at $x = 0$ (i.e., the Maclaurin series!)

Solution: Let's compute some derivatives!

$$\begin{aligned}
f(x) &= \cos x & & f(0) = 1 \\
f'(x) &= -\sin x & & f'(0) = 0 \\
f''(x) &= -\cos x & \Rightarrow \quad & f''(0) = -1 \\
f'''(x) &= \sin x & & f'''(0) = 0 \\
f^{(4)}(x) &= \cos x & & f^{(4)}(0) = 1 \\
&\vdots & & \vdots
\end{aligned}$$

(now it repeats!)

So the Maclaurin series is

$$\cos x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \boxed{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots}$$

or, in sigma form:

$$\boxed{\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}}$$

<u>Ex:</u> Find the Maclaurin series for $f(x) = \sin x$.

<u>Solution:</u> We could compute $f^{(n)}(0)$ as we did with $\cos x$, but alternatively, we can integrate $\cos x$ and its Maclaurin series:

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

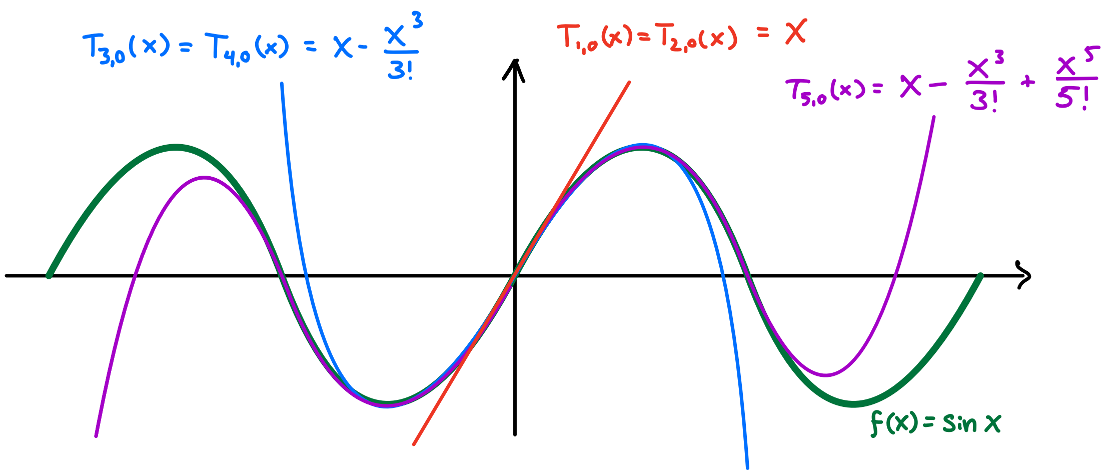
$$\Rightarrow \int \cos x \, dx = \int \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \right) dx$$

$$\Rightarrow \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} + C$$

When $x=0$, we get $\sin(0) = \cancel{\sum_{n=0}^{\infty} \frac{(-1)^n 0^{2n+1}}{(2n+1)!}}^{=0} + C$, hence

$C = \sin 0 = 0$. Thus,

$$\boxed{\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots}$$



The Taylor polynomials for $\sin x$ are becoming better and better approximations to $\sin x$, even far from $x=0$!

With infinitely many terms, $\sin x$ is <u>equal</u> to its Taylor series <u>everywhere</u>!

Takeaway: The following functions are equal to their Maclaurin series on the given interval:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad x \in (-1,1)$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad x \in (-\infty, \infty)$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad x \in (-\infty, \infty)$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad x \in (-\infty, \infty)$$

We can manipulate these series to obtain Taylor / Maclaurin series for a variety of other functions.

Ex: Find the Taylor series for $f(x) = e^x$ centred at $x = 3$.

Solution: Two options.

1. Use the definition of the Taylor series.

$$e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(3)}{n!}(x-3)^n = \boxed{\sum_{n=0}^{\infty} \frac{e^3}{n!}(x-3)^n}$$

$f^{(n)}(x) = e^x$ for all n,

so $f^{(n)}(3) = e^3$ for all n.

② Use manipulations of known series.

$$e^x = e^{(x-3)+3} = e^3 e^{x-3}$$

$$= e^3 \sum_{n=0}^{\infty} \frac{(x-3)^n}{n!}$$

Replace x with $x-3$ in the Maclaurin series for e^x.

$$= \boxed{\sum_{n=0}^{\infty} \frac{e^3}{n!}(x-3)^n}$$

Ex: Find the Taylor series for $\sin x$ centred at $x = \frac{\pi}{2}$.

Solution: We have

$$\sin x = \sin\left(\left(x - \frac{\pi}{2}\right) + \frac{\pi}{2}\right)$$

$$= \sin\left(x - \frac{\pi}{2}\right)\underbrace{\cos\frac{\pi}{2}}_{=0} + \cos\left(x - \frac{\pi}{2}\right)\underbrace{\sin\frac{\pi}{2}}_{=1}$$

$$= \cos\left(x - \frac{\pi}{2}\right)$$

Replace x with $x - \frac{\pi}{2}$ in the Maclaurin series for $\cos x$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n \left(x - \frac{\pi}{2}\right)^{2n}}{(2n)!}}$$

Ex: Find the Maclaurin series for $g(x) = x^7 \cos(x^3)$.

Solution: Calculating derivatives of $g(x)$ would be awful, so we will not use the definition of the Maclaurin series. Instead,

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\Rightarrow \quad \cos(x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n}}{(2n)!}$$

$$\Rightarrow \quad x^7 \cos(x^3) = x^7 \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n}}{(2n)!} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+7}}{(2n)!}}$$

Ex: What is the value of

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \cdots \, ?$$

Solution:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!}\Bigg|_{x=-1} = e^x\Big|_{x=-1} = \boxed{e^{-1}}$$

Ex: What is the value of $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{9^n (2n)!}$?

Solution:

$$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{9^n (2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n (\pi/3)^{2n}}{(2n)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}\Bigg|_{x=\frac{\pi}{3}}$$

$$= \cos(\pi/3)$$

$$= \boxed{1/2}$$