§ 4.2 – Separable Differential Equations

A first order DE is said to be separable if it can be written as

$$\frac{dy}{dx} = g(x) \cdot h(y)$$

That is, we can factor the non-derivative terms as a product of an x-part and a y-part.

e.g., $\frac{dy}{dx} = \frac{x}{y} \quad \left(= x \cdot \frac{1}{y}\right) \Rightarrow$ Separable

$\frac{dy}{dx} = x + 2y \, , \quad \frac{dy}{dx} = \sin(xy) \Rightarrow$ Non-separable

To solve: Split up the differential, separate the x's and y's, and integrate!

$$\frac{dy}{dx} = g(x)\,h(y) \Rightarrow \frac{1}{h(y)}\,dy = g(x)\,dx$$

$$\Rightarrow \int \frac{1}{h(y)}\,dy = \int g(x)\,dx$$

Ex: Solve $\frac{dy}{dx} = \frac{x}{y}$

Solution: $\frac{dy}{dx} = \frac{x}{y} \Rightarrow y\,dy = x\,dx$

$$\Rightarrow \int y\,dy = \int x\,dx$$

Now isolate for y (if possible!)

$$\Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + C$$

$$\Rightarrow y^2 = x^2 + 2C$$

For convenience, let's write $D = 2C$.

$$\Rightarrow \boxed{y = \pm\sqrt{x^2 + D}, \quad D \in \mathbb{R}}$$

Note: When $D=1$ we get $y = \sqrt{x^2+1}$, which is the solution we verified in our last example!

Ex: Find the general solution to $\frac{dy}{dx} = \frac{x \ln x}{3y^2}$

Solution: $\int 3y^2\,dy = \int x \ln x\,dx$

IBP!

$u = \ln x$, $du = \frac{1}{x}dx$ | $v = \frac{x^2}{2}$, $dv = x\,dx$

$$\Rightarrow y^3 = \frac{x^2}{2}\ln x - \int \frac{x^{\not 2}}{2}\cdot\frac{1}{\not x}\,dx$$

$$\Rightarrow y^3 = \frac{x^2}{2}\ln x - \frac{x^2}{4} + C$$

$$\Rightarrow \boxed{y = \sqrt[3]{\frac{x^2}{2}\ln x - \frac{x^2}{4} + C}}$$

Ex: Find the general solution to $\frac{dy}{dx} = \frac{y\cos x}{1+2y^2}$.

(Note: You won't be able to isolate for y here!)

Solution: $\int \frac{1+2y^2}{y}\,dy = \int \cos x\,dx$ ⊛

$$\Rightarrow \int \left(\frac{1}{y} + 2y\right) dy = \sin x + C$$

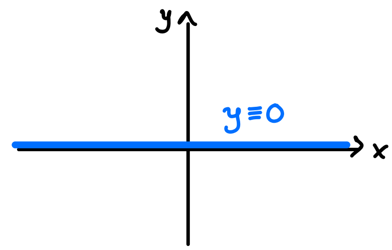
$$\Rightarrow \ln|y| + y^2 = \sin x + C$$

Can't solve for $y = f(x)$, so we'll stop here.

BUT WAIT — there may be another solution!

We divided by y in the first step ⊛, but what if $y \equiv 0$?

The notation "$y \equiv 0$" means "y is identically / constantly 0."



We'll need to check the $y \equiv 0$ case separately!

If $y \equiv 0$ then $\frac{dy}{dx} = 0$ and $\frac{y\cos x}{1+2y^2} = \frac{0 \cdot \cos x}{1+0} = 0$

So yes, $\frac{dy}{dx} = \frac{y\cos x}{1+2y^2}$ when $y \equiv 0$!

General Solution:

$$y \equiv 0 \quad \text{or} \quad \ln|y| + y^2 = \sin x + C, \quad C \in \mathbb{R}$$

If we are told the value of the unknown function y at some given x, we can figure out the arbitrary constant. A DE together with point (x_0, y_0) is called an initial value problem (IVP).

Ex: Solve the IVP $y' = xy$, $y(0) = 2$.

Solution: $\frac{dy}{dx} = xy \Rightarrow \frac{dy}{y} = x\,dx$ if $y \not\equiv 0$

(but in this case $y(0) = 2$, so $y \equiv 0$ isn't possible!)

$$\Rightarrow \int \frac{dy}{y} = \int x\,dx$$

$$\Rightarrow \ln|y| = \frac{x^2}{2} + C$$

$$\Rightarrow |y| = e^{\frac{x^2}{2} + C} = e^C e^{\frac{x^2}{2}}$$

$$\Rightarrow y = \pm e^C \cdot e^{\frac{x^2}{2}} = A e^{\frac{x^2}{2}}$$

For convenience, write $A = \pm e^C$

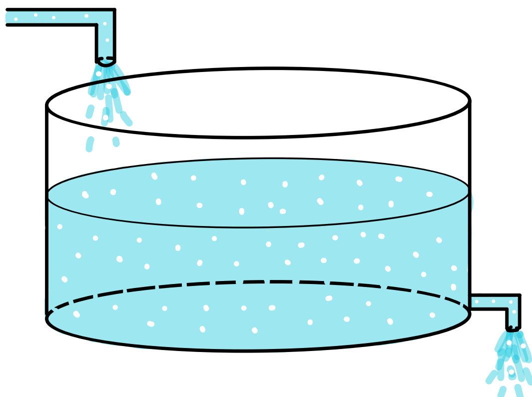
Using the initial condition $y(0) = 2$, we have

$$2 = A e^{\frac{0^2}{2}} = A \cdot 1 \Rightarrow A = 2.$$

Thus, $$\boxed{y = 2e^{\frac{x^2}{2}}}$$

Let's see a more applied example!

Ex: A tank contains 1000L of salt water at a concentration of 0.5 kg/L. Salt water at concentration 0.2 kg/L flows into the tank at a rate of 10 L/min, is thoroughly mixed, and then flows out at the same rate.



Determine the amount of salt in the tank at time t.

Solution: Let A(t) denote the amount of salt in the tank at time t. We have

$$\frac{dA}{dt} = \text{rate of salt in} - \text{rate of salt out}$$

Here,

$$\text{rate of salt in} = \underbrace{\left(0.2\ \text{kg}/\text{L}\right)}_{\text{concentration}}\underbrace{\left(10\ \text{L}/\text{min}\right)}_{\text{flow}} = 2\ \text{kg}/\text{min}$$

For the rate of salt out, note that as the new solution is mixed into the tank, we have

$$\text{concentration at time } t = \frac{\text{amount of salt at time } t}{\text{volume at time } t}$$

$$= \frac{A(t)}{1000} \leftarrow \text{volume is constant!}$$

Hence,

$$\text{rate of salt out} = \underbrace{\left(\frac{A}{1000}\ \text{kg}/\text{L}\right)}_{\text{concentration}}\underbrace{\left(10\ \text{L}/\text{min}\right)}_{\text{flow}} = \frac{A}{100}\ \text{kg}/\text{min}$$

We therefore solve the (separable) DE

$$\frac{dA}{dt} = 2 - \frac{A}{100} = -\frac{1}{100}(A-200).$$

We get

$$\int \frac{1}{A-200}\,dA = \int \frac{-1}{100}\,dt$$

$$\Rightarrow \quad \ln|A-200| = -\frac{t}{100} + C$$

$$\Rightarrow \quad |A-200| = e^{-t/100 + C} = e^{-t/100} e^{C}$$

$$\Rightarrow \quad A = 200 \pm e^{C} e^{-t/100}$$

($\pm e^{C} = B$)

$$\Rightarrow \quad A = 200 + Be^{-t/100}$$

Initially, the tank contains $0.5\ \text{kg/L} \cdot 1000\ \text{L} = 500\ \text{kg}$ of salt, hence $A(0) = 500$. This gives

$$500 = 200 + Be^{0} = 200 + B \quad \Rightarrow \quad \underline{B = 300}$$

$$\boxed{\therefore\ A(t) = 200 + 300e^{-t/100}}$$

Additional Exercises

1. Find the general solution to $\frac{dy}{dx} = y^2$.

2. Solve the IVP $\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2y - 2}$, $y(0) = -1$

Solutions

We'll separately check $y \equiv 0$

1. $$\frac{dy}{dx} = y^2 \Rightarrow \int \frac{1}{y^2}\, dy = \int 1 \cdot dx$$

$$\Rightarrow \frac{-1}{y} = x + C$$

$$\Rightarrow \frac{1}{y} = -x - C$$

Let $D = -C$

$$\Rightarrow y = \frac{1}{D - x}, \quad D \in \mathbb{R}$$

Check $y \equiv 0$: $\frac{dy}{dx} = 0$, $y^2 = 0$ (equal!)

$\Rightarrow y \equiv 0$ is a solution!

Thus,

$$\boxed{y \equiv 0 \text{ or } y = \frac{1}{D - x}, \quad D \in \mathbb{R}}$$

2. $\dfrac{dy}{dx} = \dfrac{3x^2+4x+2}{2y-2} \Rightarrow \int (2y-2)\,dy = \int (3x^2+4x+2)\,dx$

$$\Rightarrow \quad y^2 - 2y = x^3 + 2x^2 + 2x + C$$

Since $y(0) = -1$, we have

$$\underbrace{(-1)^2 - 2(-1)}_{=3} = \underbrace{0^3 + 2(0)^2 + 2(0)}_{=0} + C,$$

and hence $C = 3$. Thus,

$$y^2 - 2y = x^3 + 2x^2 + 2x + 3.$$

Solve for y by completing the square:

$$(y^2 - 2y + 1) - 1 = x^3 + 2x^2 + 2x + 3$$

$$\Rightarrow \quad (y-1)^2 = 1 + (x^3 + 2x^2 + 2x + 3)$$

$$\Rightarrow \quad y - 1 = \pm\sqrt{x^3 + 2x^2 + 2x + 4}$$

$$\Rightarrow \quad y = 1 \pm \sqrt{x^3 + 2x^2 + 2x + 4}$$

However, only

$$y = 1 - \sqrt{x^3 + 2x^2 + 2x + 4}$$

satisfies the initial condition $y(0) = -1$.