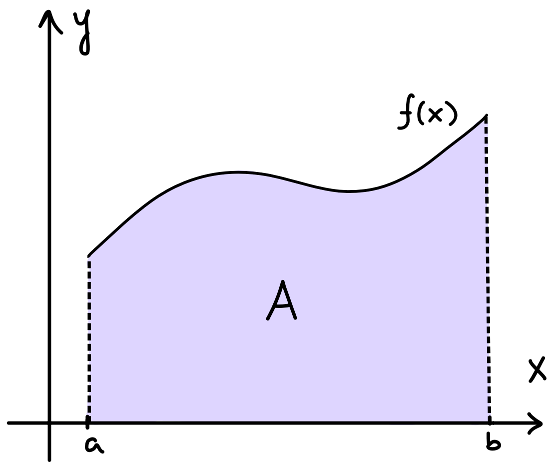
Welcome to MATH 138!

§1.2 - The Definite Integral

Goal: Calculate the area under the graph of $y = f(x)$ and above the x-axis from $x = a$ to $x = b$.



Idea: Divide the region into rectangles and add their areas to approximate A.

Definition: A partition P for the interval $[a, b]$ is a finite sequence of increasing numbers of the form

$$a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b$$

The partition divides $[a, b]$ into n subintervals

$$[t_0, t_1], [t_0, t_1], \ldots, [t_{n-2}, t_{n-1}], [t_{n-1}, t_n],$$

which may not all have the same length.

We define

$$\Delta t_i = \text{length of } [t_{i-1}, t_i] = t_i - t_{i-1}$$

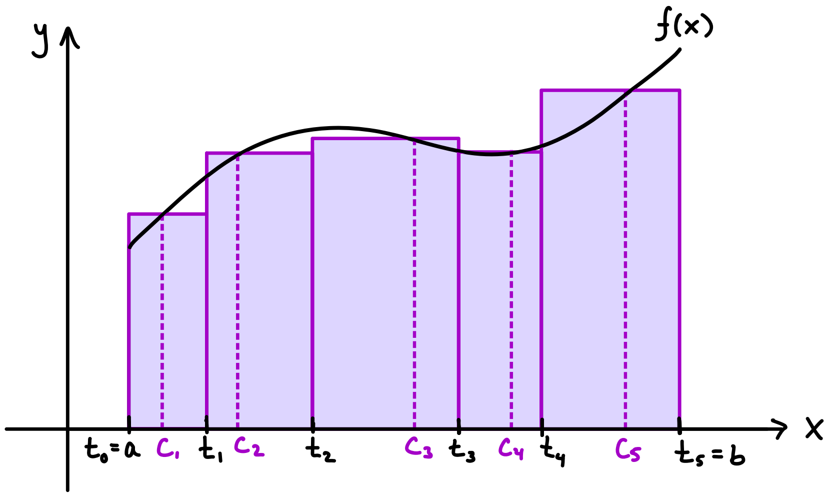
and refer to the length of the widest subinterval as the norm of the partition:

$$\| P \| = \max \{ \Delta t_1, \Delta t_2, \ldots, \Delta t_n \}.$$

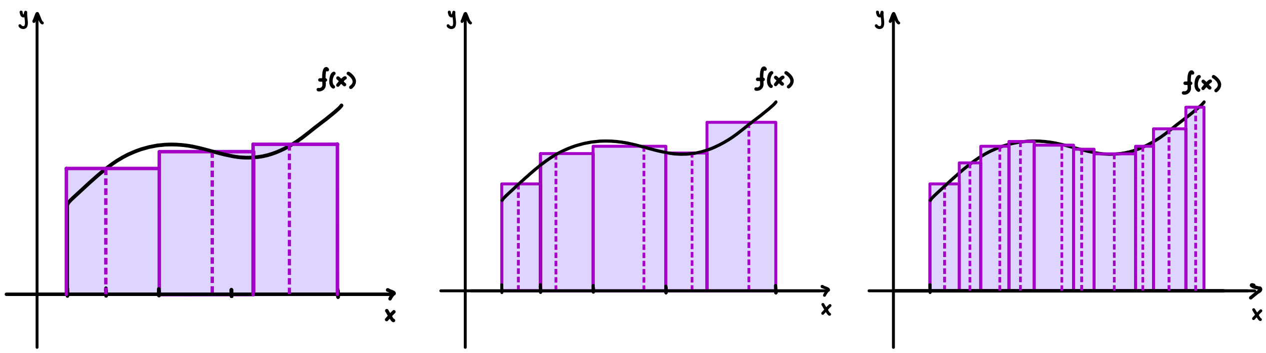
<u>Definition:</u> Given a bounded function f on $[a,b]$, a partition P of $[a,b]$, and a set of points $\{c_1, c_2, \ldots, c_n\}$ with $c_i \in [t_{i-1}, t_i]$, then a Riemann sum for f with respect to P is

$$S = f(c_1)\Delta t_1 + f(c_2)\Delta t_2 + \cdots + f(c_n)\Delta t_n = \sum_{i=1}^{n} f(c_i)\Delta t_i$$

Rectangle areas



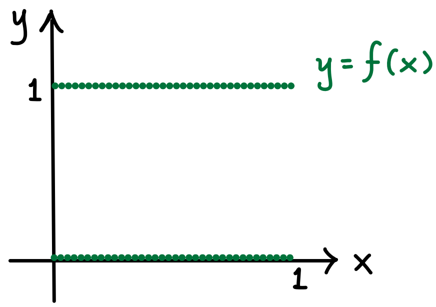
<u>Idea:</u> To make the approximation exact, consider a sequence of partitions $\{P_n\}$ with $\|P_n\| \longrightarrow 0$ and compute $\lim\limits_{n\to\infty} S_n$, where $\{S_n\}$ is a sequence of Riemann sums corresponding to the P_n's



<u>The Issue:</u> For some nasty functions, the value of this limit may depend on how we select our c_i's !

Example: For $x \in [0,1]$, define

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational,} \\ 1 & \text{if } x \text{ is irrational.} \end{cases}$$



If we consider a sequence of Riemann sums where all c_i are rational, then $\sum_{i=1}^{n} \underbrace{f(c_i)}_{=0} \Delta t_i = 0$, hence $\lim_{n\to\infty} S_n = 0$.

But if instead all c_i's were irrational we would have

$$\sum_{i=1}^{n} \underbrace{f(c_i)}_{=1} \Delta t_i = \sum_{i=1}^{n} \Delta t_i = 1 \quad \text{(the length of } [0,1]\text{)},$$

hence $\lim_{n\to\infty} S_n = 1$ (different!).

Such functions don't have a well-defined area.

Let's instead focus on the nicer functions that do!

Definition: We say that f is <u>integrable</u> if there exists a unique number $I \in \mathbb{R}$ such that, if whenever $\{P_n\}$

is a sequence of partitions with $\lim_{n\to\infty} \|P_n\| = 0$ and $\{S_n\}$ is any sequence of Riemann sums associated to the P_n's, we have

$$\lim_{n\to\infty} S_n = I.$$

In this case we call I the <u>definite integral</u> of f over $[a,b]$ and denote it

Variable of integration

Bounds of integration

$$\int_a^b f(t)\,dt$$

Integrand

<u>Note:</u> The name of the variable doesn't affect the value of the integral: $\int_a^b f(t)\,dt = \int_a^b f(x)\,dx = \int_a^b f(z)\,dz$

So... what types of functions are integrable?

<u>Theorem</u> (Integrability of Continuous Functions): If f is continuous on $[a,b]$, then f is integrable on $[a,b]$.

Note: The theorem also holds for functions with only finitely many discontinuities.

Thus, to compute $\int_a^b f(t)\,dt$ when f is continuous, we can use any sequence of partitions with $\|P_n\| \longrightarrow 0$ and any associated sequence of Riemann sums (since all will produce the same result). Let's pick some simple ones!

The regular n partition of [a, b]:

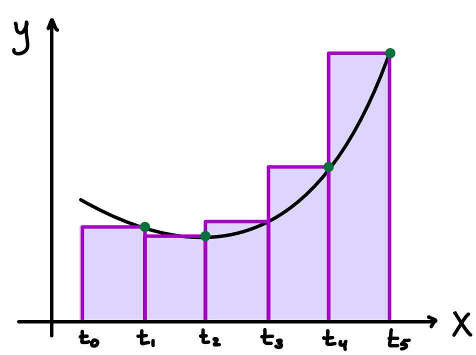
All subintervals have equal width $\Delta t = \dfrac{b-a}{n}$.

In this case, $t_i = a + i\,\Delta t$

The right endpoint Riemann sum:

$$S_n = \sum_{i=1}^{n} f(t_i)\,\Delta t = \sum_{i=1}^{n} f(a + i\Delta t)\,\Delta t$$

$c_i = t_i$, the right endpoint of $[t_{i-1}, t_i]$



Thus, if f is continuous and we use the regular n partitions and right endpoint Riemann sums, we get

$$\boxed{\int_a^b f(t)\,dt = \lim_{n\to\infty} \sum_{i=1}^{n} f(a+i\Delta t)\,\Delta t, \qquad \Delta t = \frac{b-a}{n}}$$

Ex: Calculate $\int_0^2 (4x^3 - x)\,dx$. Note that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$.

Solution: With $f(x) = 4x^3 - x$, we have

$$\int_0^2 (4x^3 - x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(\underbrace{0}_{a=0} + i\Delta x)\,\underbrace{\Delta x}_{\Delta x = \frac{b-a}{n} = \frac{2-0}{n}}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\cdot\frac{2}{n}$$

$$= \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} \left[4\left(\frac{2i}{n}\right)^3 - \left(\frac{2i}{n}\right)\right]$$

$$= \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} \left[\frac{32i^3}{n^3} - \frac{2i}{n}\right]$$

$$= \lim_{n\to\infty} \left[\frac{64}{n^4} \sum_{i=1}^{n} i^3 - \frac{4}{n^2} \sum_{i=1}^{n} i \right]$$

$$= \lim_{n\to\infty} \left[\frac{64}{\cancel{n^4}} \cdot \frac{\cancel{n^2}(n+1)^2}{4} - \frac{4}{\cancel{n^2}} \cdot \frac{\cancel{n}(n+1)}{2} \right]$$

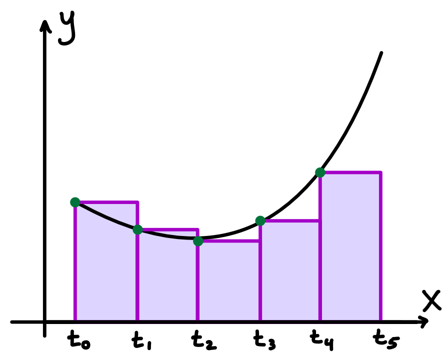
$$= \lim_{n\to\infty} \left[\underbrace{\frac{16(n^2+2n+1)}{n^2}}_{\to 16} - \underbrace{\frac{2(n+1)}{n}}_{\to 2} \right] = \boxed{14}$$

Alternatively, we could have used...

Left endpoint Riemann sums:

$$S_n = \sum_{i=1}^{n} f(t_{i-1})\,\Delta t = \sum_{i=1}^{n} f\big(a+(i-1)\Delta t\big)\,\Delta t$$

$c_i = t_{i-1}$, the left endpoint of $[t_{i-1}, t_i]$



in which case we could compute the integral as

$$\int_a^b f(t)\,dt = \lim_{n\to\infty} \sum_{i=1}^{n} f\big(a+(i-1)\Delta t\big)\Delta t, \qquad \Delta t = \frac{b-a}{n}$$

Exercise: Estimate $\int_0^1 x^2\,dx$ with $n = 3$ left endpoint rectangles from a regular partition of $[0,1]$.

Solution: $\Delta x = \frac{b-a}{n} = \frac{1-0}{3} = \frac{1}{3}$, hence the Riemann sum is

$$\sum_{i=1}^{3} f(0 + (i-1)\Delta x)\,\Delta x = \sum_{i=1}^{3} \left(\frac{i-1}{3}\right)^2 \cdot \frac{1}{3}$$

$$= 0^2 \cdot \frac{1}{3} + \left(\frac{1}{3}\right)^2 \cdot \frac{1}{3} + \left(\frac{2}{3}\right)^2 \cdot \frac{1}{3}$$

$$= \boxed{\frac{5}{27}}$$

Exercise: Calculate $\int_0^1 x^2\,dx$ exactly, given that

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Solution: We have $\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$. Using right endpoint Riemann sums, we have

$$\int_0^1 x^2\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(0 + i\Delta x)\,\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{i}{n}\right) \cdot \frac{1}{n}$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2$$

$$= \lim_{n\to\infty} \frac{1}{n^3} \sum_{i=1}^{n} i^2$$

$$= \lim_{n\to\infty} \frac{1}{\cancel{n^3}} \cdot \frac{\cancel{n}(n+1)(2n+1)}{6}$$

$$= \lim_{n\to\infty} \frac{2n^2+3n+1}{6n^2} = \frac{2}{6} = \boxed{\frac{1}{3}}$$

The result will be the same if we use left endpoints

$$\int_0^1 x^2\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(0+(i-1)\Delta x)\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i-1}{n}\right)^2 \cdot \frac{1}{n} \qquad \text{(try it!)}$$

but using right endpoints is often a bit simpler.