§6.5, 6.6 - Review of Taylor Polynomials

We can approximate a function f around $x=a$ using a Taylor polynomial!

Definition: If f is n-times differentiable at $x=a$, then the nth-degree Taylor polynomial for $f(x)$ centred at $x=a$ is

$$T_{n,a}(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Ex: Compute $T_{1,1}(x)$ and $T_{2,1}(x)$ for $f(x)=\sqrt{x}$.

Solution:

$$f(x) = \sqrt{x} \qquad\qquad f(0) = 1$$

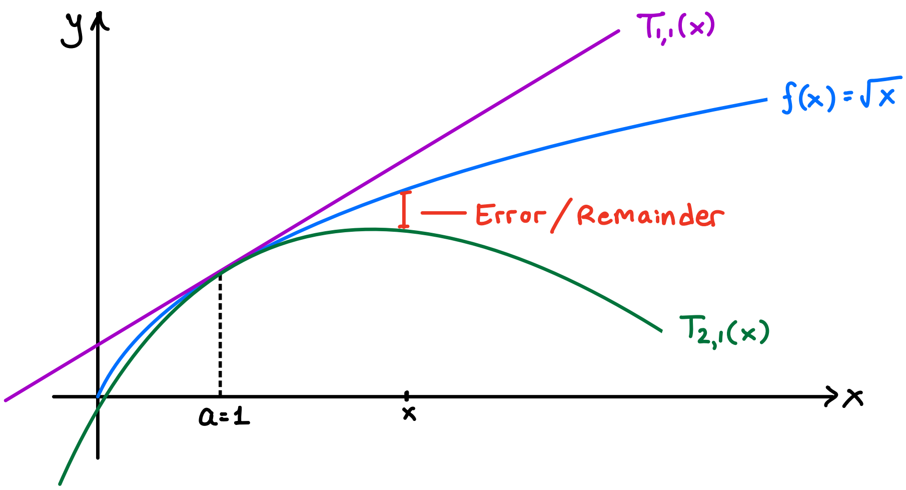
$$f'(x) = \frac{1}{2\sqrt{x}} \quad \Rightarrow \quad f'(0) = \frac{1}{2}$$

$$f''(x) = \frac{-1}{4x^{3/2}} \qquad\qquad f''(0) = \frac{-1}{4}$$

So,

$$T_{1,1}(x) = f(1) + f'(1)(x-1) = \underline{1 + \tfrac{1}{2}(x-1)}$$

$$T_{2,1}(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 = \underline{1 + \tfrac{1}{2}(x-1) - \tfrac{1}{8}(x-1)^2}$$



The error or remainder in approximating $f(x)$ using $T_{n,a}(x)$ is defined to be

$$R_{n,a}(x) = f(x) - T_{n,a}(x).$$

Taylor's Theorem: Suppose f is $(n+1)$-times differentiable throughout an interval I containing a. For every

$x \in I$, the error in approximating $f(x)$ with $T_{n,a}(x)$ has the form

$$\boxed{R_{n,a}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}}$$

for some c between a and x.

As a consequence, we obtain Taylor's Inequality, which allows us to estimate the size of the error!

Taylor's Inequality: Suppose f is $(n+1)$-times differentiable throughout an interval I containing a. If $x \in I$ and $|f^{(n+1)}(c)| \leq M$ for all c between a and x, then

$$\boxed{|R_{n,a}(x)| \leq \frac{M|x-a|^{n+1}}{(n+1)!}}$$

We will make use of this bound in the next section!