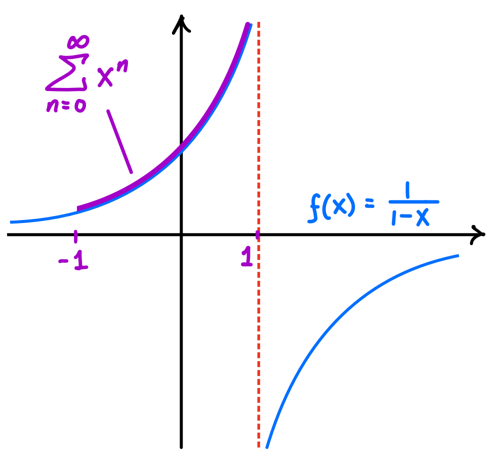
# §6.2 – Representing Functions as Power Series

**Recall:** A power series is a function whose domain is its interval of convergence. Sometimes we can recognize it as a more familiar function!

**Ex:** From geometric series, we know that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots \quad \text{for } |x| < 1 \quad (\text{i.e., } I = (-1,1),\ R = 1)$$




According to the following theorem, power series always turn out to be continuous, hence $\sum_{n=0}^{\infty} x^n$ can't possibly represent $\frac{1}{1-x}$ across its vertical asymptote at $x = 1$.

**Theorem (Abel):** If $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ with interval of convergence $I$, then $f$ is continuous on $I$.

If we know a power series representation for a function, we have some tricks for obtaining power series representations for related functions.

Indeed, suppose $f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$ and $g(x) = \sum_{n=0}^{\infty} b_n (x-a)^n$ with radii of convergence $R_f$ and $R_g$ and intervals of convergence $I_f$ and $I_g$, respectively.

## 1. Addition / Subtraction

$$\boxed{f(x) \pm g(x) = \sum_{n=0}^{\infty} \left(a_n \pm b_n\right)(x-a)^n.}$$

If $R_f \neq R_g$, then the new series has radius of convergence $R = \min\{R_f, R_g\}$ and interval of convergence $I = I_f \cap I_g$. If $R_f = R_g$, then $R \geq R_f$.

(e.g., $f(x) = g(x) = \sum_{n=0}^{\infty} x^n$ have radii $R_f = R_g = 1$, but $f(x) - g(x) = \sum_{n=0}^{\infty} 0 \cdot x^n = 0$ has radius $R = \infty$.)

## 2. Multiplication

For $k = 1, 2, 3, \ldots,$

$$\boxed{(x-a)^k f(x) = \sum_{n=0}^{\infty} c_n (x-a)^{n+k}}$$

and the new series has the same radius of convergence,

$R_f$ and interval of convergence $I_f$

## 3. Composition

If $a = 0$ $\left(\text{so } f(x) = \sum_{n=0}^{\infty} a_n x^n\right)$, then for $c \in \mathbb{R}$ and

$k = 1, 2, 3, \ldots$

$$\boxed{f(cx^k) = \sum_{n=0}^{\infty} a_n c^k x^{nk}}$$

We can also replace $x$ with $x - b$ to change to obtain

a power series for $f(x-b)$ centred at $x = b$:

$$\boxed{f(x-b) = \sum_{n=0}^{\infty} a_n (x-b)^n}$$

Compositions can change the radius and interval of convergence, as we will see in our examples.

Ex: Find a power series representation centred at $x = 0$ for $f(x) = \frac{1}{3-x}$

Solution: We know that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$,

$$\Rightarrow \frac{1}{3-x} = \frac{1}{3}\left(\frac{1}{1-\frac{x}{3}}\right) = \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n = \boxed{\sum_{n=0}^{\infty}\frac{x^n}{3^{n+1}}}$$

Replace $x$ with $\frac{x}{3}$ in $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

For convergence, we need $\left|\frac{x}{3}\right| < 1 \Rightarrow |x| < 3$

$\therefore$ Radius of convergence is $R = 3$.

Interval of convergence is $I = (-3, 3)$.

Ex: Find a power series representation centred at $x=0$ for $f(x) = \frac{x^2}{x+7}$.

**Solution:** Again, $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$, so

$$\frac{x^2}{x+7} = \frac{x^2}{7}\left(\frac{1}{1+\frac{x}{7}}\right) = \frac{x^2}{7}\left(\frac{1}{1-\left(\frac{-x}{7}\right)}\right)$$

$$= \frac{x^2}{7}\sum_{n=0}^{\infty}\left(\frac{-x}{7}\right)^n$$

Converges when $\left|\frac{-x}{7}\right| < 1$, hence $|x| < 7$.

$$= \frac{x^2}{7}\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{7^n}$$

$$= \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n x^{n+2}}{7^{n+1}}}$$

and this series has radius of convergence $\underline{R = 7}$ and interval of convergence $\underline{I = (-7, 7)}$.

**Ex:** Find a power series representation centred at $x = 2$ for $f(x) = \frac{1}{x}$.

**Solution:** We're looking for a power series of the form

$$\frac{1}{x} = \sum_{n=0}^{\infty} c_n (x-2)^n$$

To introduce powers of $x-2$, the trick is to add and subract 2.

$$\frac{1}{x} = \frac{1}{(x-2)+2} = \frac{1}{2}\left[\frac{1}{1+\frac{x-2}{2}}\right]$$

$$= \frac{1}{2}\left[\frac{1}{1-\left(\frac{-(x-2)}{2}\right)}\right]$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{-(x-2)}{2}\right)^n$$

$$= \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n(x-2)^n}{2^{n+1}}}$$

This converges when $\left|\frac{-(x-2)}{2}\right| < 1$, hence $|x-2| < 2$.

Thus, $-2 < x-2 < 2$, or equivalently, $0 < x < 4$.

Radius of convergence: $\underline{R = 2}$

Interval of convergence: $\underline{I = (0,4)}$