§5.9, 5.10 – The Ratio and Root Tests

Our final two convergence tests can, in some cases, tell us that a series converges absolutely.

The Ratio Test

Consider a series $\sum a_n$ and suppose $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$ exists or is ∞.

(i) If $L < 1$, $\sum a_n$ converges absolutely

(ii) If $L > 1$, $\sum a_n$ diverges

(iii) If $L = 1$, the test is inconclusive. The series could converge absolutely, conditionally, or diverge.

Remark: The ratio test is often effective when dealing with factorials: We define $0! = 1$ and

$$n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1 \qquad (n \in \mathbb{N})$$

e.g., $5! = 5\cdot 4\cdot 3\cdot 2\cdot 1 = 120$

Ex: Determine whether each series below converges absolutely, converges conditionally, or diverges.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \cdot 3^n}{n!}$

Solution: We'll use the ratio test!

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{\cancel{(-1)}^{n+1} \cdot 3^{n+1}}{(n+1)!}}{\frac{\cancel{(-1)}^{n} \cdot 3^{n}}{n!}} \right|$$

$$= \lim_{n\to\infty} \frac{3^{n+1}}{3^n} \frac{n!}{(n+1)!} \qquad = \frac{n!}{(n+1)\cdot n!} = \frac{1}{n+}$$

$$= \lim_{n\to\infty} 3 \cdot \frac{1}{n+1} = 0$$

Since $L < 1$, $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 3^n}{n!}$ converges absolutely by the ratio test.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{9^n}{n \cdot 2^{n+1}}$

Solution: Again, let's try the ratio test!

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{9^{n+1}}{(n+1)\cdot 2^{n+2}}}{\frac{9^n}{n\cdot 2^{n+1}}} \right|$$

$$= \lim_{n\to\infty} \frac{9^{n+1}}{9^n} \cdot \frac{2^{n+1}}{2^{n+2}} \cdot \frac{n}{n+1}$$

$$= \lim_{n\to\infty} \frac{9}{2}\left(\frac{n}{n+1}\right) \overset{LH}{=} \frac{9}{2}$$

Since $L > 1$, $\sum_{n=1}^{\infty} \frac{9^n}{n\cdot 2^{n+1}}$ diverges by the ratio test.

(c) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2 \cdot \ln(n)}$

Solution: Maybe the ratio test will work again?

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{\cancel{(-1)}^{n+1}}{(n+1)^2 \ln(n+1)}}{\frac{\cancel{(-1)}^n}{n^2 \cdot \ln(n)}} \right|$$

$$= \lim_{n\to\infty} \frac{n^2}{(n+1)^2} \cdot \frac{\ln(n)}{\ln(n+1)}$$

$$= \underbrace{\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^2}_{\overset{LH}{=}\left(\lim_{n\to\infty}\frac{1}{1}\right)^2 = 1} \cdot \underbrace{\lim_{n\to\infty}\frac{\ln(n)}{\ln(n+1)}}_{\overset{LH}{=}\lim_{n\to\infty}\frac{1/n}{1/(n+1)}} = \underline{1}$$

$$= \lim_{n\to\infty}\frac{n+1}{n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1$$

Uh oh... if $L = 1$, the ratio test is inconclusive!

To determine if $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2\cdot\ln(n)}$ converges absolutely,

we'll need to examine

$$\sum_{n=2}^{\infty}\left|\frac{(-1)^n}{n^2\cdot\ln(n)}\right| = \sum_{n=2}^{\infty}\frac{1}{n^2\cdot\ln(n)}.$$

Note that $\displaystyle\frac{1}{n^2\cdot\ln(n)} \le \frac{1}{n^2}$ and $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n^2}$ is a

convergent p-series (as $p = 2 > 1$). Thus, $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n^2\ln(n)}$

converges, so $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2\cdot\ln(n)}$ <u>converges absolutely</u>.

The Root Test

Consider a series $\sum a_n$ and suppose $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$ exists or is ∞.

(i) If $L < 1$, $\sum a_n$ converges absolutely

(ii) If $L > 1$, $\sum a_n$ diverges

(iii) If $L = 1$, the test is inconclusive. The series could converge absolutely, conditionally, or diverge.

Remark: The root test is often effective when dealing with series involving expressions like $(f(n))^n$.

Ex: Apply the root test to $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n+1}{3n+1}\right)^n$.

Solution:

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$$

$$= \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{n+1}{3n+1}\right)^n\right|}$$

$$= \lim_{n\to\infty} \frac{n+1}{3n+1}$$

$$= \frac{1}{3}$$

Since $L < 1$, the series converges absolutely by the root test.

Ex: Apply the root test to $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n e^{n^2}}{n^n}$.

Solution: $$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left| \frac{\cancel{(-1)^n} e^{n^2}}{n^n} \right|}$$

$$= \lim_{n\to\infty} \left(\frac{e^{n^2}}{n^n} \right)^{1/n}$$

$$= \lim_{n\to\infty} \frac{e^n}{n}$$

$$\overset{LH}{=} \lim_{n\to\infty} \frac{e^n}{1} = \infty.$$

Thus, since $L > 1$, the series diverges by the root test.

Additional Exercise

Ex: Let $a_n, b_n \neq 0$ for all n. Prove that if $\sum b_n$ converges and $\sum \frac{a_{n+1}}{a_n b_n}$ converges absolutely, then $\sum a_n$ converges absolutely.

Solution: Since $\sum \left|\frac{a_{n+1}}{a_n b_n}\right|$ converges, we have $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n b_n}\right| = 0$ by the divergence test. This means that for n sufficiently large, we have $\left|\frac{a_{n+1}}{a_n b_n}\right| < 1$, or equivalently

$$0 \leq \left|\frac{a_{n+1}}{a_n}\right| < |b_n|$$

Since $\sum b_n$ converges, $\lim_{n\to\infty} b_n = 0$ by the divergence test, and hence

$$\lim_{n\to\infty} |b_n| = \left|\lim_{n\to\infty} b_n\right| = 0.$$

By the squeeze theorem,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0$$

Thus, $\sum a_n$ converges absolutely by the ratio test. $\blacksquare$