§1.3 – Properties of the Integral

Theorem: Let f and g be integrable on $[a,b]$.

(i) For any $c \in \mathbb{R}$, $\displaystyle\int_a^b c\, f(x)\,dx = c\int_a^b f(x)\,dx$

(ii) $$\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

(iii) If $m \leq f(x) \leq M$, for $x \in [a,b]$, then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

(iv) If $f(x) \geqslant 0$, then $\displaystyle\int_a^b f(x)\,dx \geq 0$

(v) If $f(x) \leq g(x)$ for all $x \in [a,b]$, then

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$$

(vi) $|f|$ is integrable on $[a,b]$ and

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$$

Proofs: (i), (ii): follow from the limit laws for sequences.

(iii): Consider the regular right endpoint Riemann sums

$$\sum_{i=1}^{n} f(t_i)\,\Delta t$$

Since $m \leq f(t_i) \leq M$ for all i, we have

$$\sum_{i=1}^{n} m\,\Delta t \leq \sum_{i=1}^{n} f(t_i)\,\Delta t \leq \sum_{i=1}^{n} M\,\Delta t$$

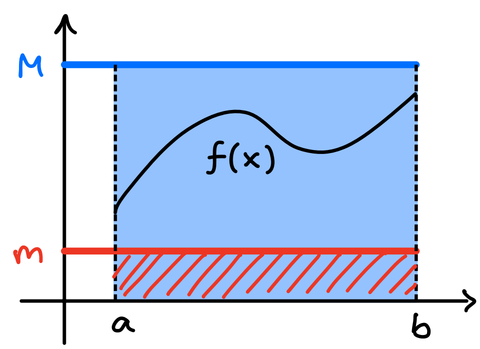
$$\Rightarrow \quad m \underbrace{\sum_{i=1}^{n} \Delta t}_{=\text{ length of } [a,b]} \leq \sum_{i=1}^{n} f(t_i)\,\Delta t \leq M \underbrace{\sum_{i=1}^{n} \Delta t}_{=\text{ length of } [a,b]}$$

$$\Rightarrow \quad m(b-a) \leq \sum_{i=1}^{n} f(t_i)\,\Delta t \leq M(b-a)$$

Taking limits as $n \to \infty$, we get

$$m(b-a) \leq \int_a^b f(t)\,dt \leq M(b-a)$$

Basically, (iii) says that f encloses more than the red area but less than the blue



(iv): follows from (iii) with $m = 0$.

(v): If $f(x) \leq g(x)$, then $0 \leq g(x) - f(x)$. Hence,

$$\underset{\text{By (iv)}}{0 \leq} \int_a^b g(x) - f(x)\, dx \underset{\text{By (i), (ii)}}{=} \int_a^b g(x)\, dx - \int_a^b f(x)\, dx$$

(Move to the LHS!)

Thus, $$\int_a^b f(x)\, dx \leq \int_a^b g(x)\, dx$$

(vi): Follows from the triangle inequality. ■

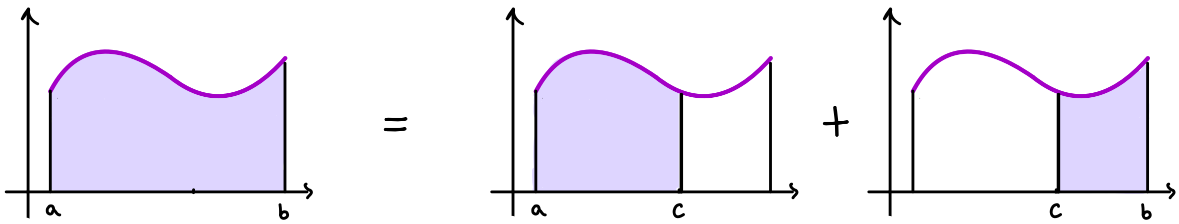
<u>Additional Properties</u>: Suppose f is integrable on an interval containing a, b, and c. We have

(i) $$\int_a^a f(x)\, dx = 0$$ (This is really a definition!)

(ii) $$\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$$

(iii) $$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$$

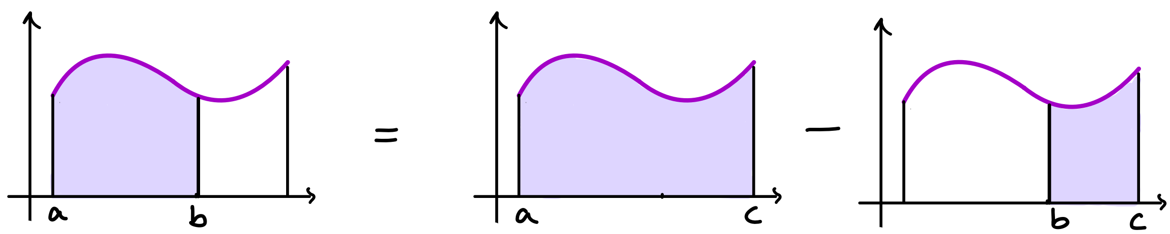
You can think of (iii) in terms of areas:



$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

and it even works when c isn't between a and b!

For instance, if $a < b < c$, then



$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx - \int_b^c f(x)\,dx$$

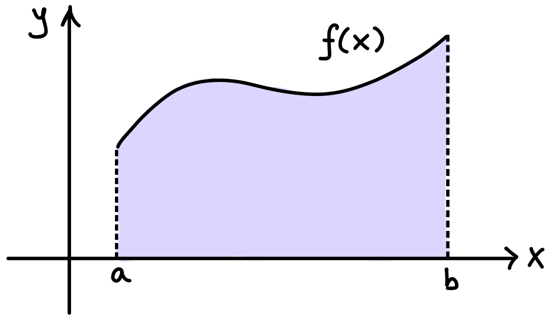
Flip the bounds using (ii)

$$\Rightarrow \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

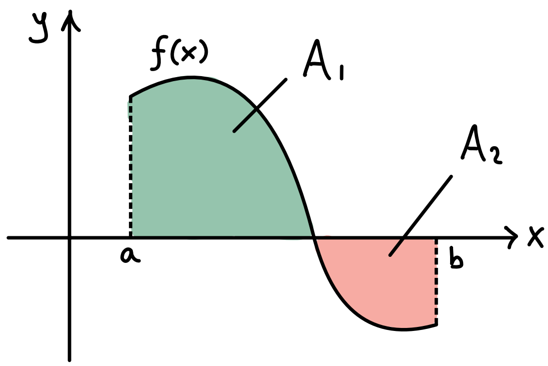
Geometric Interpretation of the Integral

Suppose f is integrable on $[a, b]$.

If $f(x) \geqslant 0$ for all $x \in [a,b]$, then $\int_a^b f(x)\,dx$ represents the area under the graph of f and above the x-axis.



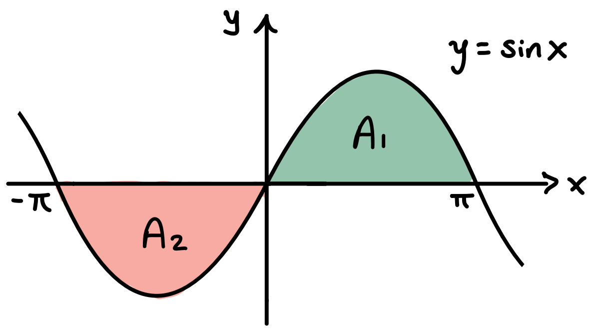
More generally, $\int_a^b f(x)\,dx$ represents the signed area between the graph of f and the x-axis with area below the x-axis counted negatively



$$\int_a^b f(x)\,dx = A_1 - A_2$$

Example: What is $\int_{-\pi}^{\pi} \sin x \, dx$?

<u>Solution:</u> This is probably too complicated to do with Riemann sums. However, since $\int_{-\pi}^{\pi} \sin x \, dx$ is the signed area between $y = \sin x$ and the x-axis ...



... we will get $\int_{-\pi}^{\pi} \sin x \, dx = A_1 - A_2 = 0$, since $\sin x$ is odd and hence symmetric about the origin.