§6.1 – Power Series

Definition: A series of the form

$$\sum_{n=0}^{\infty} \underbrace{c_n}_{\text{coefficients depending on } n} (\underbrace{x}_{\text{variable}}-\underbrace{a}_{\text{constant}})^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

is called a power series centred at $x = a$.

A power series is really a function, just expressed as an infinite series. Its domain is all $x \in \mathbb{R}$ such that the series converges.

Remark: A power series will always converge at its centre, $x = a$, since

$$x = a \Rightarrow \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1\underbrace{(a-a)}_{=0} + c_2\underbrace{(a-a)^2}_{=0} + \cdots$$

$$= c_0 \quad (\text{finite!})$$

But what if we plug in other x's? Will the series converge if we plug in $x=1$? $x=0.5$? Not sure!

We can answer this question using the ratio test!

Ex: For which x does the power series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$ converge?

Solution: Given $x \in \mathbb{R}$, we use the ratio test:

$$L = \lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| = \lim_{n\to\infty} \frac{n!}{(n+1)!} \cdot \frac{|x|^{n+1}}{|x|^n}$$

$$= \lim_{n\to\infty} \frac{|x|}{n+1} = 0$$

Since $L<1$ always, $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges (absolutely) for all $x \in (-\infty, \infty)$.

Ex: For which values of x does the power series

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n \cdot 2^n}$$ converge?

Solution: Using the ratio test, we compute

$$L = \lim_{n\to\infty} \left| \frac{\frac{(x-3)^{n+1}}{(n+1)\,2^{n+1}}}{\frac{(x-3)^n}{n\cdot 2^n}} \right|$$

$$= \lim_{n\to\infty} \frac{n}{n+1} \cdot \frac{2^n}{2^{n+1}} \cdot \frac{|x-3|^{n+1}}{|x-3|^n}$$

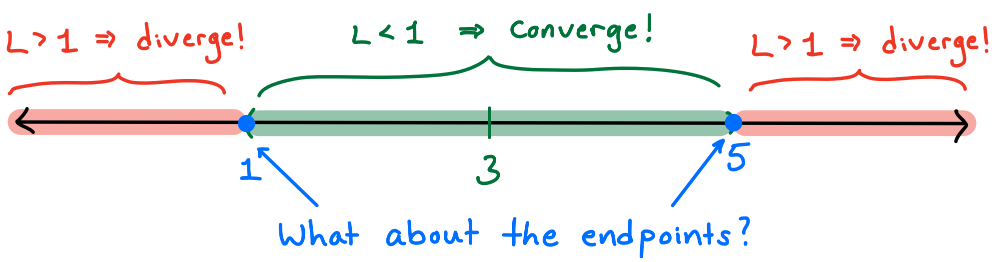
$$= \lim_{n\to\infty} \underbrace{\frac{n}{n+1}}_{\to 1} \cdot \frac{1}{2} \cdot |x-3| = \frac{|x-3|}{2}$$

The series will converge (absolutely) when $L<1$:

$$L<1 \iff \frac{|x-3|}{2} < 1 \iff \underbrace{|x-3| < 2}_{\text{"distance from } x \text{ to } 3 \text{ is } <2\text{"}}$$

And the series will diverge when $L>1$:

$$L>1 \iff \frac{|x-3|}{2} > 1 \iff \underbrace{|x-3| > 2}_{\text{"distance from } x \text{ to } 3 \text{ is } >2\text{"}}$$



At the endpoints, $x = 1$ and $x = 5$, the ratio test is inconclusive as $L = \frac{|x-3|}{2} = 1$. We need to check convergence at $x = 1$ and $x = 5$ separately using other tests.

$$x = 5 \Rightarrow \sum_{n=1}^{\infty} \frac{(x-3)^n}{n \cdot 2^n} = \sum_{n=1}^{\infty} \frac{(5-3)^n}{n \cdot 2^n}$$

$$= \sum_{n=1}^{\infty} \frac{\cancel{2^n}}{n \cdot \cancel{2^n}}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} \quad \text{(divergent p-series!)}$$

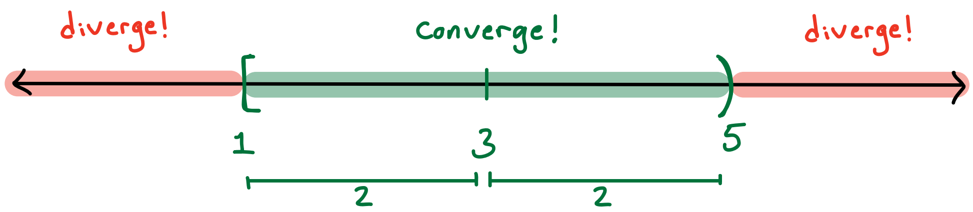
$$x = 1 \Rightarrow \sum_{n=1}^{\infty} \frac{(x-3)^n}{n \cdot 2^n} = \sum_{n=1}^{\infty} \frac{(1-3)^n}{n \cdot 2^n}$$

$$= \sum_{n=1}^{\infty} \frac{(-2)^n}{n \cdot 2^n}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n \cancel{2^n}}{n \cdot \cancel{2^n}}$$

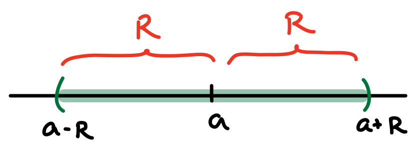
$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \quad \text{(converges by AST)}$$

Thus, $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n \cdot 2^n}$ converges for $x \in [1,5)$.

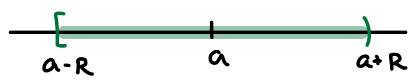


Theorem: A power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ will always converge on an interval I centred at $x=a$ and will diverge outside of I.

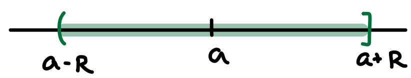
$I = (a-R, a+R)$



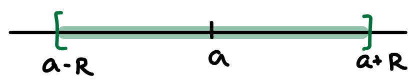
or $I = [a-R, a+R)$



or $I = (a-R, a+R]$



or $I = [a-R, a+R]$



We call I the interval of convergence and call R the radius of convergence.

Using our new terminology, we can say

- $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ (1st example) has interval of convergence $I = (-\infty, \infty)$ and radius of convergence $R = \infty$

- $\sum_{n=0}^{\infty} \frac{(x-3)^n}{n \cdot 2^n}$ (2nd example) has interval of convergence $I = [1, 5)$ and radius of convergence $R = 2$

The distance from the centre, $a = 3$, to the edge of the interval $[1, 5)$.

Remarks:

1. To find the radius and interval of convergence, use the ratio test and determine where $L < 1$. Convergence

at the endpoints of I must be checked separately using other tests.

2. If $x \in I$ is NOT an endpoint of I, convergence at x will be absolute. If $x \in I$ is an endpoint, convergence at x could be conditional or absolute.

Ex: Find the radius and interval of convergence.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (x-5)^n}{n^2 \cdot 3^n}$ (b) $\displaystyle\sum_{n=0}^{\infty} n!\,(x+1)^n$

Solutions:

(a) We use the ratio test:

$$L = \lim_{n\to\infty} \left| \frac{\dfrac{\cancel{(-1)^{n+1}}(x-5)^{n+1}}{(n+1)^2 \cdot 3^{n+1}}}{\dfrac{\cancel{(-1)^n}(x-5)^n}{n^2 \cdot 3^n}} \right| = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} \cdot \frac{3^n}{3^{n+1}} \cdot \frac{|x-5|^{n+1}}{|x-5|^n}$$

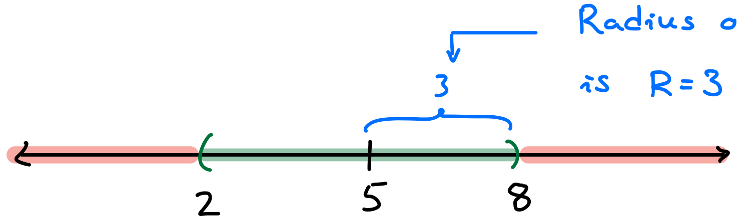
$$= \lim_{n\to\infty} \underbrace{\left(\frac{n}{n+1}\right)^2}_{\to 1^2} \cdot \frac{|x-5|}{3}$$

$$= \frac{|x-5|}{3}$$

We have

$$L < 1 \iff \frac{|x-5|}{3} < 1 \iff |x-5| < 3$$

Radius of convergence is $R = 3$



We must check convergence at the endpoints!

$$x = 8 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n (x-5)^n}{n^2 \cdot 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n (8-5)^n}{n^2 \cdot 3^n}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n \cdot \cancel{3^n}}{n^2 \cdot \cancel{3^n}}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \quad \text{(converges by AST)}$$

$$x = 2 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n (x-5)^n}{n^2\, 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n (2-5)^n}{n^2\, 3^n}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n (-3)^n}{n^2 \cdot 3^n}$$

$$= \sum_{n=1}^{\infty} \frac{\overbrace{(-1)^n [(-1)^n \cancel{3^n}]}^{=1}}{n^2 \cancel{3^n}}$$

Thus,

$$= \sum_{n=1}^{\infty} \frac{1}{n^2} \quad \text{(convergent p-series)}$$

$$\boxed{I = [2, 8] \text{ and } R = 3}$$

(b) Using the ratio test, we have

$$L = \lim_{n\to\infty} \left| \frac{(n+1)!\,(x+1)^{n+1}}{n!\,(x+1)^n} \right| = \lim_{n\to\infty} (n+1)\,|x+1|$$

$$= \infty \quad \text{for all } x \neq -1.$$

Since $L > 1$ for all $x \neq -1$, the series diverges for all such x and hence only converges at its centre, $x = -1$. Thus,

$$\boxed{I = \{-1\}, \quad R = 0.}$$