§2.3 – Partial Fractions

This technique is useful for evaluating $\int \frac{P(x)}{Q(x)}\,dx$,

where P, Q are polynomials and

Important assumption!

$$\boxed{\text{degree of } P(x) < \text{degree of } Q(x)}$$

Motivating Example: $\int \frac{3x+1}{x^2+2x-3}\,dx$.

Looks tough... but there's a trick! We can write

(check this!)

$$\frac{3x+1}{x^2+2x-3} = \frac{3x+1}{(x-1)(x+3)} = \frac{1}{x-1} + \frac{2}{x+3}$$

This is called a partial fraction decomposition (PFD)

and it makes integration much easier! Indeed,

Since $\int \frac{1}{x+a}\,dx = \ln|x+a| + C$, we have

(Exercise! Let $u = x+a$.)

$$\int \frac{3x+1}{x^2+2x-3}\,dx = \int \left(\frac{1}{x-1} + \frac{2}{x+3}\right) dx$$

$$= \underline{\ln|x-1| + 2\ln|x+3| + C}$$

Cool! But how can we find partial fraction decompositions ourselves??

Step 1: Fully factor the denominator into linear terms $(ax+b)$ and irreducible quadratic terms.

ax^2+bx+c is irreducible if it has no real roots (equivalently, if $b^2-4ac<0$)

e.g. $\dfrac{1}{x^3+2x^2+x} = \dfrac{1}{x(x^2+2x+1)} = \dfrac{1}{x(x+1)^2}$

linear

linear (repeated)

$\dfrac{x+2}{x^4+4x^2} = \dfrac{x+2}{x^2(x^2+4)}$

linear (repeated)

irreducible quadratic

Step 2: Write down the form of the PFD.

Distinct linear factors each get a constant numerator in the PFD, and repeated linear factors get one constant per power.

constants (to be determined)

e.g.

$$\frac{1}{(x+2)(x+4)} = \frac{A}{x+2} + \frac{B}{x+4}$$

$$\frac{x+1}{(x+2)(x+4)} = \frac{C}{x+2} + \frac{D}{x+4}$$

The form of the PFD depends only on the factors in the denominator.

The numerator will only affect the values of the constants.

one per power   one per power

$$\frac{2}{x^3(x+1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x+1} + \frac{E}{(x+1)^2}$$

Distinct irreducible quadratics each get a linear numerator $(Ax+B)$ in the PFD. Repeated irreducible quadratics get a linear numerator per power.

e.g.

$$\frac{1}{(x+2)(x^2+4)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+4}$$

$$\frac{x+3}{x(x^2+1)(x^2+x+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{x^2+x+4}$$

$$\frac{x^2+x+1}{(x+4)^2(x^2+9)^3} = \frac{A}{x+4} + \frac{B}{(x+4)^2} + \frac{Cx+D}{x^2+9} + \frac{Ex+F}{(x^2+9)^2} + \frac{Gx+H}{(x^2+9)^3}$$

<u>Step 3:</u> Solve for the constants A, B, C, etc...

Let's see an example of this!

<u>Ex:</u> Find the PFD for $\frac{3x+1}{x^2+2x-3}$.

<u>Solution:</u> $\frac{3x+1}{x^2+2x-3} = \frac{3x+1}{(x-1)(x-3)} = \frac{A}{x-1} + \frac{B}{x+3}$.

Multiply both sides by the denominator of the LHS:

$$3x+1 = (x-1)(x+3)\left(\frac{A}{x-1} + \frac{B}{x+3}\right)$$

$$\Rightarrow \quad 3x+1 = A(x+3) + B(x-1)$$

<u>Option 1:</u> Solve for A and B by plugging in some nice values for x.

$$x = 1 \Rightarrow 4 = A\cdot(1+3) + B(1-1) = 4A \Rightarrow \underline{A = 1}$$

$$x = -3 \Rightarrow -8 = A(-3+3) + B(-3-1) = -4B \Rightarrow \underline{B = 2}$$

$$\therefore \frac{3x+1}{x^2+2x-3} = \frac{A}{x-1} + \frac{B}{x+3} = \boxed{\frac{1}{x-1} + \frac{2}{x-3}}$$

<u>Option 2</u>: Solve for A and B by comparing coefficients!

$$3x+1 = A(x+3) + B(x-1) \Rightarrow \boxed{3}x + \boxed{1} = \boxed{(A+B)}x + \boxed{(3A-B)}$$

$$\Rightarrow \begin{cases} 3 = A+B \\ 1 = 3A-B \end{cases}$$

Solve the system!

$$\Rightarrow \underline{A = 1, \quad B = 2}$$

<u>Ex</u>: Calculate $\int \frac{x^2+x+8}{x^3+4x}\,dx$ using a PFD.

<u>Solution</u>: $\frac{x^2+x+8}{x^3+4x} = \frac{x^2+x+8}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$

$$\Rightarrow \quad x^2+x+8 = x(x^2+4)\left[\frac{A}{x} + \frac{Bx+C}{x^2+4}\right]$$

$$\Rightarrow \quad x^2+x+8 = A(x^2+4) + (Bx+C)x$$

$$\Rightarrow \quad x^2+x+8 = (A+B)x^2 + Cx + 4A$$

Comparing coefficients: $\begin{cases} 1 = A+B \\ 1 = C \\ 8 = 4A \end{cases} \Rightarrow A=2, \ B=-1, \ C=1$

Thus, $\dfrac{x^2+x+8}{x^3+4x} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+4} = \boxed{\dfrac{2}{x} + \dfrac{-x+1}{x^2+4}}$

Let's now evaluate $\displaystyle\int \frac{x^2+x+8}{x^3+4x}\,dx$!

$$\int \frac{x^2+x+8}{x^3+4x}\,dx = \underbrace{\int \frac{2}{x}\,dx}_{2\ln|x|} + \underbrace{\int \frac{-x+1}{x^2+4}\,dx}_{\text{Harder... let's split up the integral!}}$$

$$= 2\ln|x| - \underbrace{\int \frac{x}{x^2+4}\,dx}_{\substack{u\text{-sub! } u=x^2+4 \\ du = 2x\,dx}} + \underbrace{\int \frac{1}{x^2+4}\,dx}_{\substack{\text{trig sub! } x=2\tan\theta \\ dx = 2\sec^2\theta\, d\theta}}$$

$$= 2\ln|x| - \int \frac{\cancel{x}}{u} \cdot \frac{du}{2\cancel{x}} + \int \frac{1}{4\tan^2\theta + 4} \cdot 2\sec^2\theta \, d\theta$$

$$= 2\ln|x| - \frac{1}{2}\int \frac{1}{u}\,du + \int \frac{2\cancel{\sec^2\theta}}{4\cancel{(\underbrace{\tan^2\theta + 1}_{\sec^2\theta})}}\,d\theta$$

$$= 2\ln|x| - \frac{1}{2}\ln|u| + \frac{1}{2}\int 1\,d\theta$$

$$= 2\ln|x| - \frac{1}{2}\ln|x^2+4| + \frac{1}{2}\theta + C \qquad \left(\begin{array}{l} x = 2\tan\theta \\ \Rightarrow \theta = \arctan\left(\frac{x}{2}\right) \end{array}\right)$$

$$= \boxed{2\ln|x| - \frac{1}{2}\ln|x^2+4| + \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C}$$

Ex: Evaluate $\displaystyle\int \frac{x^3 - 2x}{x^2 + 2x + 1}\,dx$

Don't use partial fractions yet! If the degree of the numerator is greater than or equal to the degree of the denominator, start with long division!

Aside: Polynomial Long Division

Ex: Consider the division $\dfrac{x^3-2x}{x^2+2x+1}$

First, set up the long division.

$$x^2+2x+1 \overline{)\, x^3-2x}$$

Next, what do we need to multiply the largest term in the denominator by to match the largest term in the numerator? Write the answer at the top.

$$\begin{array}{r} x \\ \underline{x^2}+2x+1 \overline{)\, \underline{x^3}-2x} \end{array}$$

Multiply the denominator by this answer and subtract the result from the numerator.

multiply!

$$\begin{array}{rl} & \; x \\ x^2+2x+1 & \overline{)\, x^3 \qquad\quad -2x} \\ & \underline{-(x^3+2x^2+x)} \\ & \qquad -2x^2-3x \end{array}$$

Repeat the process until you obtain a polynomial with smaller degree than your denominator.

$$
\begin{array}{r}
x-2 \\
x^2+2x+1 \,\big)\, \overline{x^3 \qquad -2x} \\
-(x^3+2x^2+x) \\
\hline
-2x^2-3x \\
-(-2x^2-4x-2) \\
\hline
x+2
\end{array}
$$

The polynomial at the top is the quotient. The polynomial at the bottom is the remainder.

$$
\begin{array}{r}
\overbrace{x-2}^{\text{Quotient}} \\
x^2+2x+1 \,\big)\, \overline{x^3 \qquad -2x} \\
-(x^3+2x^2+x) \\
\hline
-2x^2-3x \\
-(-2x^2-4x-2) \\
\hline
\underbrace{x+2}_{\text{Remainder}}
\end{array}
$$

Answer:

$$\frac{x^3-2x}{x^2+2x+1} = \overbrace{x-2}^{\text{Quotient.}} + \frac{\overbrace{x+2}^{\text{Remainder}}}{x^2+2x+1}$$

Easy to integrate

Use partial fractions!

We have

$$\frac{x+2}{x^2+2x+1} = \frac{x+2}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2}$$

$$\Rightarrow \quad x+2 = A(x+1) + B$$

$$\Rightarrow \quad A = 1, \; B = 1$$

Thus,

$$\frac{x^3-2x}{x^2+2x+1} = x-2+\frac{x+2}{x^2+2x+1}$$

$$= x-2+\left(\frac{1}{x+1}+\frac{1}{(x+1)^2}\right)$$

$$\Rightarrow \int \frac{x^3-2x}{x^2+2x+1}\,dx = \int (x-2)\,dx + \int \frac{1}{x+1}\,dx + \int \frac{1}{(x+1)^2}\,dx$$

$u = x+1$

$du = dx$

$$= \frac{x^2}{2} - 2x + \ln|x+1| + \int \frac{1}{u^2}\,du$$

$$= \frac{x^2}{2} - 2x + \ln|x+1| - \frac{1}{u} + C$$

$$= \boxed{\frac{x^2}{2} - 2x + \ln|x+1| - \frac{1}{x+1} + C}$$

Additional Exercises:

1. Evaluate each integral below.

(a) $\displaystyle\int \frac{x}{x^2-4x-5}\,dx$ (b) $\displaystyle\int \frac{5x+8}{x^3+4x^2+4x}\,dx$ (c) $\displaystyle\int \frac{2x^2-5x-4}{2x+1}\,dx$

2. Integrate each function given its PFD below:

(a) $$\frac{x^3-2}{(x^2+2x+2)(x+2)^2} = \frac{1}{x^2+2x+2} + \frac{1}{x+2} - \frac{5}{(x+2)^2}$$

(b) $$\frac{x^2-3x+9}{(x^2+9)^3} = \frac{1}{(x^2+9)^2} - \frac{3x}{(x^2+9)^3}$$

Solutions

1. (a) The PFD of the function is

$$\frac{x}{x^2-4x-5} = \frac{x}{(x-5)(x+1)} = \frac{A}{x-5} + \frac{B}{x+1}.$$

$$\Rightarrow \quad x = (x-5)(x+1)\left(\frac{A}{x-5} + \frac{B}{x+1}\right)$$

$$\Rightarrow \quad x = A(x+1) + B(x-5)$$

$$x = 5 \quad \Rightarrow \quad 5 = A\cdot(5+1) + B(\cancel{5-5}) = 6A \quad \Rightarrow \quad A = 5/6$$

$$x = -1 \quad \Rightarrow \quad -1 = A(\cancel{-1+1}) + B(-1-5) = -6B \quad \Rightarrow \quad B = 1/6$$

$$\therefore \quad \frac{x}{x^2-4x+5} = \frac{A}{x-5} + \frac{B}{x+1} = \frac{5/6}{x-5} + \frac{1/6}{x+1}$$

$$\Rightarrow \int \frac{x}{x^2-4x+5}\,dx = \frac{5}{6}\int \frac{1}{x-5}\,dx + \frac{1}{6}\int \frac{1}{x+1}\,dx$$

$$= \boxed{\frac{5}{6}\ln|x-5| + \frac{1}{6}\ln|x+1| + C.}$$

(b) The PFD of the function is

$$\frac{5x+8}{x^3+4x^2+4x} = \frac{5x+8}{x(x+2)^2} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$

$$\Rightarrow 5x+8 = x(x+2)^2\left(\frac{A}{x} + \frac{B}{x+2} + \frac{C}{(x+2)^2}\right)$$

$$\Rightarrow 5x+8 = A(x+2)^2 + Bx(x+2) + Cx$$

When $x=0$: $5(0)+8 = A(0+2)^2 + 0 + 0$

$\Rightarrow 8 = 4A$

$\Rightarrow \underline{A = 2}$

When $x=-2$: $5(-2)+8 = 0+0+C(-2)$

$\Rightarrow -2 = -2C$

$\Rightarrow \underline{C = 1}$

When $x=1$: $5(1)+8 = \underbrace{A\cdot 3^2}_{=18} + B\cdot 1\cdot 3 + \underbrace{C\cdot 1}_{=1}$

Plug in anything else to find B

$\Rightarrow 13 = 19 + 3B$

$\Rightarrow -6 = 3B$

$\Rightarrow \underline{B = -2}$

Our PFD is $\dfrac{5x+8}{x^3+4x^2+4x} = \dfrac{2}{x} - \dfrac{2}{x+2} + \dfrac{1}{(x+2)^2}$

$$\int \frac{5x+8}{x^3+4x^2+4x}\,dx = \int \left(\frac{2}{x} - \frac{2}{x+2} + \frac{1}{(x+2)^2}\right) dx$$

$$= 2\ln|x| - 2\ln|x+2| + \int \frac{1}{(x+2)^2}\,dx$$

let $u = x+2$

$$= 2\ln|x| - 2\ln|x+2| + \int u^{-2}\,du$$

$$= \boxed{2\ln|x| - 2\ln|x+2| - \frac{1}{x+2} + D}$$

(c) We'll start with long division.

$$\begin{array}{r} x-3 \\ 2x+1 \overline{)\ 2x^2-5x-4} \\ -(2x^2+x) \\ \hline -6x-4 \\ -(-6x-3) \\ \hline -1 \end{array}$$

Thus, $\dfrac{2x^2-5x-4}{2x+1} = x-3 - \dfrac{1}{2x+1}$

$$\Rightarrow \int \frac{2x^2-5x-4}{2x+1}\,dx = \int (x-3)\,dx - \int \frac{1}{2x+1}\,dx$$

$u = 2x+1$

$du = 2dx$

$$= \frac{x^2}{2} - 3x - \frac{1}{2}\int \frac{1}{u}\,du$$

No partial fractions needed!

$$= \boxed{\frac{x^2}{2} - 3x - \frac{1}{2}\ln|2x+1| + C}$$

2(a)

$$\int \frac{x^3-2}{(x^2+2x+2)(x+2)^2}\,dx = \underbrace{\int \frac{1}{x^2+2x+2}\,dx}_{\text{Complete the square!}} + \underbrace{\int \frac{1}{x+2}\,dx}_{=\ln|x+2|} - 5\underbrace{\int \frac{1}{(x+2)^2}\,dx}_{\substack{u\text{-sub: } u=x+2 \\ du=dx}}$$

$$= \underbrace{\int \frac{1}{(x+1)^2+1}\,dx}_{\substack{\text{trig sub: } x+1=\tan\theta \\ dx=\sec^2\theta\,d\theta}} + \ln|x+2| - 5\int \frac{1}{u^2}\,du$$

$$= \int \frac{\sec^2\theta}{\underbrace{\tan^2\theta+1}_{=\sec^2\theta}}\,d\theta + \ln|x+2| - 5\left[\frac{u^{-1}}{-1}\right]$$

$$= \int 1\,d\theta + \ln|x+2| + \frac{5}{u}$$

$$= \theta + \ln|x+2| + \frac{5}{x+2} + C$$

$$= \boxed{\arctan(x+1) + \ln|x+2| + \frac{5}{x+2} + C}$$

(b) $$\int \frac{x^2-3x+9}{(x^2+9)^3}\,dx = \underbrace{\int \frac{1}{(x^2+9)^2}\,dx}_{\substack{\text{trig sub: } x = 3\tan\theta \\ dx = 3\sec^2\theta\, d\theta}} - \underbrace{\int \frac{3x}{(x^2+9)^3}\,dx}_{\substack{u\text{-sub: } u = x^2+9 \\ du = 2x\,dx \\ (\text{so } dx = du/2x)}}$$

$$= \int \frac{3\sec^2\theta}{(9\tan^2\theta+9)^2}\,d\theta - \int \frac{3\cancel{x}}{u^3}\cdot\frac{du}{2\cancel{x}}$$

$$= \int \frac{3\sec^2\theta}{9^2(\underbrace{\tan^2\theta+1}_{\sec^2\theta})^2}\,d\theta - \frac{3}{2}\int u^{-3}\,du$$

$$= \frac{1}{27}\int \frac{\sec^2\theta}{\sec^4\theta}\,d\theta - \frac{3}{2}\cdot\frac{u^{-2}}{-2} + C$$

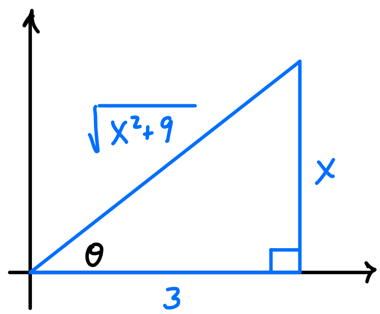
$$= \frac{1}{27}\int \cos^2\theta\,d\theta + \frac{3}{4(x^2+9)^2} + C$$

$$= \frac{1}{27}\int \frac{1}{2}(1+\cos 2\theta)\,d\theta$$

$$= \frac{1}{54}\left[\theta + \frac{\sin 2\theta}{2}\right] + \frac{3}{4(x^2+9)^2} + C$$

Let's convert back to x's!

$$x = 3\tan\theta \Rightarrow \tan\theta = \frac{x}{3} = \frac{\text{opposite}}{\text{adjacent}} \Rightarrow \theta = \arctan\left(\frac{x}{3}\right).$$



$$\frac{\sin 2\theta}{2} = \frac{\cancel{2}\sin\theta\cos\theta}{\cancel{2}}$$

$$= \left(\frac{x}{\sqrt{x^2+9}}\right)\left(\frac{3}{\sqrt{x^2+9}}\right)$$

$$= \frac{3x}{x^2+9}$$

$$\therefore \int \frac{x^2 - 3x + 9}{(x^2+9)^3}\,dx = \frac{1}{54}\left[\theta + \frac{\sin 2\theta}{2}\right] + \frac{3}{4(x^2+9)^2} + C$$

$$= \boxed{\frac{1}{54}\left[\text{artan}\left(\frac{x}{3}\right) + \frac{3x}{x^2+9}\right] + \frac{3}{4(x^2+9)^2} + C}$$