§4.1 - Introduction to Differential Equations

A differential equation (DE) is an equation involving an unknown function and its derivatives. The equation may contain $x, y, y', y'', \ldots, y^{(n)}$.

Typically, our goal is to solve the equation for the function(s) y.

Ex: $y' = y$ ← "Which function(s) y differentiates to itself?"

Solutions to this DE include $y = e^x$, but also $y = 0$, $y = 2e^x$, $y = \pi e^x$, ... In general, $\boxed{y = Ce^x,\ C \in \mathbb{R}}$

> The complete set of solutions to a DE (including any arbitrary constants) is called the general solution.

Definition: The order of a DE is the order of the highest derivative that appears.

Examples:

(a) $y' = y$ $\Rightarrow$ order 1

(b) $\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} = y^3$ $\Rightarrow$ order 2

(c) $\sqrt{y''} + \sin(xy) = 1$ $\Rightarrow$ order 2

In MATH 138, we'll focus mainly on DEs of order 1.

We'll learn methods for solving two types of DEs:

1. Separable DEs (next lesson!)

2. Linear DEs, which have the form

$$\boxed{A_n(x)\,y^{(n)} + A_{n-1}(x)\,y^{(n-1)} + \cdots + A_1(x)\,y' + A_0(x)\,y = C(x)}$$

n^{th} derivative of y

Examples:

(a) $y'' + \sin(x)\,y = x^3$ $\Rightarrow$ linear

(b) $y'y = e^x$ $\Rightarrow$ non-linear (due to $y'y$)

(c) $y' + y^2 = x+2$ $\Rightarrow$ non-linear (due to y^2)

Although finding solutions to DEs can be hard, it's usually easy to verify whether a specific function is a solution or not.

Ex: Is $y = \sqrt{x^2+1}$ a solution to the DE

$$\frac{dy}{dx} = \frac{x}{y}\ ?$$

Solution: Let's compute the left and right hand sides.

LHS: $$\frac{dy}{dx} = \frac{d}{dx}\sqrt{x^2+1} = \cancel{2}x \cdot \frac{1}{\cancel{2}}(x^2+1)^{-1/2} = \frac{x}{\sqrt{x^2+1}}$$

RHS: $\frac{x}{y} = \frac{x}{\sqrt{x^2+1}}$

Since LHS = RHS : $\boxed{\text{yes } y = \sqrt{x^2+1} \text{ is a solution.}}$

Direction Fields

Given a DE of the form $y' = g(x,y)$, we can actually see graphically what its solution curves will look like!

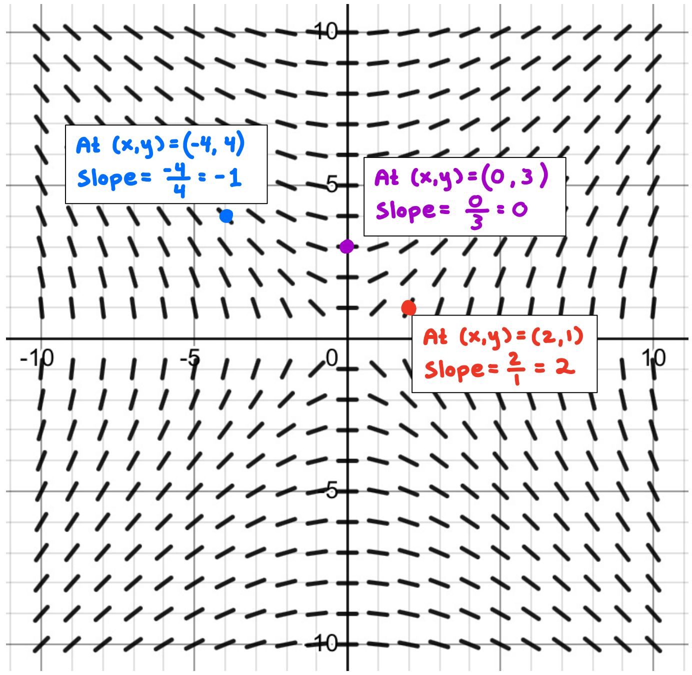
Indeed, at each point (x,y), $y' = g(x,y)$ tells us the slope of the tangent line to $y = f(x)$ at (x,y).

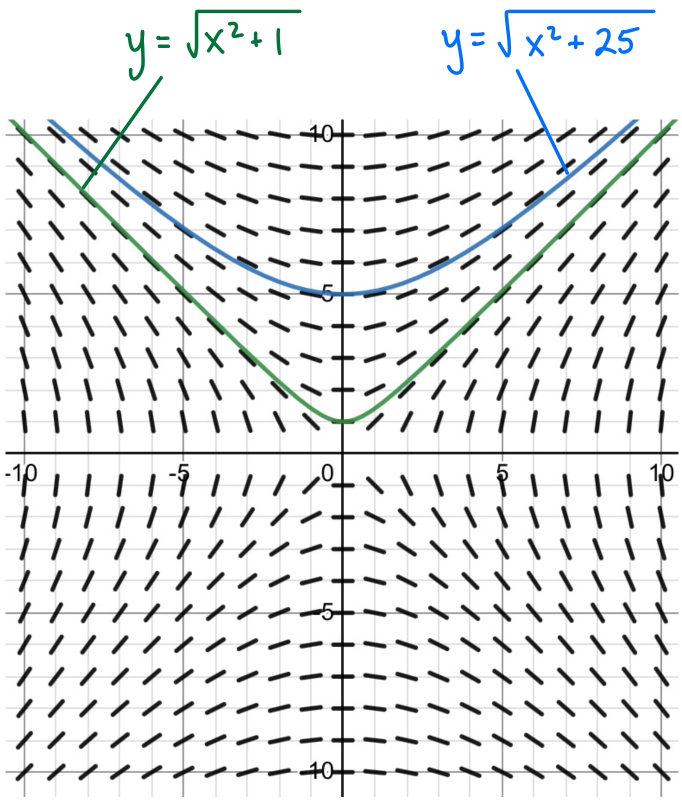
Ex: Consider the DE $y' = \frac{x}{y}$.

At each (x,y), let's plot a small line with slope $y' = \frac{x}{y}$!

We get the direction field shown on the right

(a few points have been singled out randomly.)





Two solutions to the DE $y' = \frac{x}{y}$. The sloped lines are tangent to the curves at each point (x,y).