§5.1 - Infinite Series

Let $\{a_n\}$ be a sequence. An infinite series is an expression of the form

$$\boxed{a_1 + a_2 + a_3 + a_4 + \cdots = \sum_{n=1}^{\infty} a_n}$$

Ex: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$

$\underbrace{\frac{1}{2} + \frac{1}{4}}_{\text{sum} = 0.75}$

$\underbrace{\frac{1}{2} + \frac{1}{4} + \frac{1}{8}}_{\text{sum} = 0.875}$

$\underbrace{\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}}_{\text{sum} = 0.9375}$

$\ddots$

Perhaps these terms add to 1? Hard to tell...

Let's formalize the ideas from the examples above.

Definition: Given a series $\displaystyle\sum_{n=1}^{\infty} a_n$, we define its N^{th} partial sum by

$$\boxed{S_N = a_1 + a_2 + \cdots + a_N = \sum_{n=1}^{N} a_n}$$

If $\displaystyle\lim_{N\to\infty} S_N = S$ for some (finite) real number S, then we say $\sum_{n=1}^{\infty} a_n$ converges to S and write

$$\boxed{\sum_{n=1}^{\infty} a_n = S.}$$

If instead $\lim_{N\to\infty} S_N$ DNE (i.e., doesn't approach anything or is $\pm\infty$), we say $\sum_{n=1}^{\infty} a_n$ diverges.

Ex: For $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$, we have

$S_1 = 1/2$

$S_2 = 1/2 + 1/4 = 3/4$

$S_3 = 1/2 + 1/4 + 1/8 = 7/8$

$S_4 = 1/2 + 1/4 + 1/8 + 1/16 = 15/16$ $\quad\cdots$

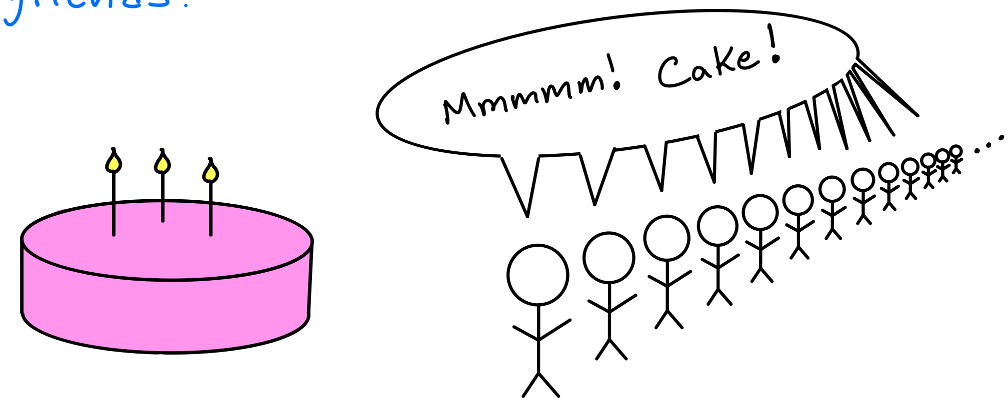
Noticing a pattern, we have $S_N = \frac{2^N - 1}{2^N}$, hence

$$\lim_{n\to\infty} S_N = \lim_{N\to\infty} \frac{2^N-1}{2^N} = \lim_{N\to\infty}\left(1 - \frac{1}{2^N}\right) = 1.$$

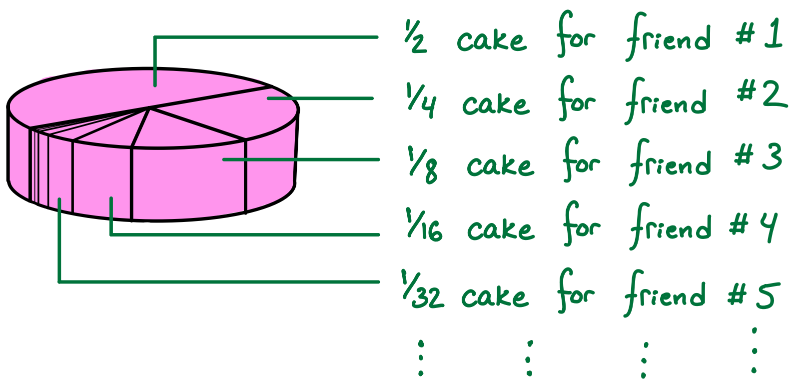
$$\boxed{\text{Thus, } \sum_{n=1}^{\infty} \frac{1}{2^n} = 1 \quad \text{(convergent!)}}$$

You can think about this particular sum as follows:

Imagine sharing a cake with your infinitely many friends.



We'll divide up the cake in the following way:



In total, $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1$ cake!

Ex: For $\sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + \cdots$, we have

$$S_0 = 1,$$

$$S_1 = 1 - 1 = 0,$$

$$S_2 = 1 - 1 + 1 = 1$$

$$S_3 = 1 - 1 + 1 - 1 = 0, \quad \ldots$$

In general, $S_N = \begin{cases} 1 & \text{if } N \text{ is even,} \\ 0 & \text{if } N \text{ is odd.} \end{cases}$

Thus, $\lim_{N\to\infty} S_N$ DNE, so $\boxed{\sum_{n=0}^{\infty} (-1)^n \text{ diverges.}}$

Ex: For $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$, we have

$$S_1 = 1 - \frac{1}{2}$$

$$S_2 = \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \frac{1}{3}\right) = 1 - \frac{1}{3}$$

$$S_3 = \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \frac{1}{4}\right) = 1 - \frac{1}{4}$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

In general,

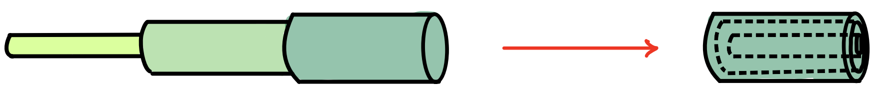
$$S_N = \left(1 - \cancel{\tfrac{1}{2}}\right) + \left(\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}}\right) + \left(\cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{4}}\right) + \cdots$$

$$\cdots + \left(\cancel{\tfrac{1}{N-1}} + \cancel{\tfrac{1}{N}}\right) + \left(\cancel{\tfrac{1}{N}} - \tfrac{1}{N+1}\right) = \boxed{1 - \tfrac{1}{N+1}}$$

Thus, $\lim\limits_{N\to\infty} S_N = \lim\limits_{N\to\infty}\left(1 - \cancelto{0}{\tfrac{1}{N+1}}\right) = 1$, hence

$$\boxed{\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 \qquad \text{(convergent!)}}$$

Note: $\sum\limits_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right)$ referred to as a telescoping series, since the middle terms "collapse" when we compute S_N!



Ex: It turns out that the harmonic series

$$\sum_{n=1}^{\infty}\frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

diverges, even though the terms $\frac{1}{n}$ are very tiny when n is large. Let's see why!

$$S_2 = 1 + \tfrac{1}{2}$$

$$S_4 = 1 + \tfrac{1}{2} + \underbrace{\tfrac{1}{3} + \tfrac{1}{4}}_{\geq \frac{1}{4} + \frac{1}{4} = \frac{1}{2}} \geq 1 + 2\left(\tfrac{1}{2}\right)$$

$$S_8 = 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{\geq \frac{1}{2}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{\geq \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}} \geq 1 + 3\left(\frac{1}{2}\right)$$

$$S_{16} = 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{\geq \frac{1}{2}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{\geq \frac{1}{2}} + \underbrace{\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16}}_{\geq \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} = \frac{1}{2}}$$

$$\geq 1 + 4\left(\tfrac{1}{2}\right)$$

$$\cdots$$

In general,

$$S_{2^N} \geq \underbrace{1 + N\left(\frac{1}{2}\right)}_{\to \infty \text{ as } N \to \infty} \text{ for all } N.$$

Thus, $\lim\limits_{N\to\infty} S_N = \infty$, meaning $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ diverges. ■

It turns out that convergent series possess many of the same properties as finite sums. The

Properties of Convergent Series

Suppose $\sum\limits_{n=1}^{\infty} a_n$ and $\sum\limits_{n=1}^{\infty} b_n$ converge with $\sum\limits_{n=1}^{\infty} a_n = A$

and $\sum_{n=1}^{\infty} b_n = B$.

1. For any $k \in \mathbb{R}$, $\sum_{n=1}^{\infty} k a_n$ converges and $\sum_{n=1}^{\infty} k a_n = kA$

2. $\sum_{n=1}^{\infty} (a_n \pm b_n)$ converges and $\sum_{n=1}^{\infty} (a_n \pm b_n) = A \pm B$

Furthermore, if $\{c_n\}_{n=1}^{\infty}$ is a sequence of real numbers and $j \in \mathbb{N}$, then

3. $\sum_{n=1}^{\infty} c_n$ converges if and only if $\sum_{n=j}^{\infty} c_n$ converges.

Note: 1. & 2. follow from rules for sums and scalar multiples of sequences, applied to the sequence of partial sums $\{S_N\}_{N=1}^{\infty}$ for $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$.

3. follows from the fact that

$$\sum_{n=1}^{\infty} c_n = \underbrace{(c_1 + c_2 + \cdots + c_{j-1})}_{\text{finite!}} + \sum_{n=j}^{\infty} c_n \,,$$

So convergence of the LHS $\left(\sum_{n=1}^{\infty} c_n\right)$ will occur exactly when $\sum_{n=j}^{\infty} c_n$ converges.

Property 3 tells us that when checking the convergence/divergence of series, we can look past (i.e., ignore) any finite number of terms.

Convergence / divergence is determined by the infinitely many terms at the end of the series!