§2.2 - Integration by Parts (IBP)

Recall: If u and v are functions of x, then by the product rule,

$$\boxed{\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}}$$

Integrating both sides, we have

$$\int \frac{d}{dx}(uv)\,dx = \int u\frac{dv}{\cancel{dx}}\,\cancel{dx} + \int v\frac{du}{\cancel{dx}}\,\cancel{dx}$$

$$\Longrightarrow \quad uv = \int u\,dv + \int v\,du$$

Rearrange

$$\Longrightarrow \quad \boxed{\int u\,dv = uv - \int v\,du.}$$

Integration by parts (IBP) formula.

Remarks:

1. IBP allows us to trade one integral, $\int u\,dv$ for another (hopefully simpler) integral, $\int v\,du$.

2. IBP can help when integrating a product of functions. One function, u, will be differentiated while the other, dv, will be integrated.

Ex: $\int x \cos x \, dx$

Strategy: Pick u to be a function that gets simpler when differentiated. Everything else is dv.

Solution: Let's define u and dv as follows:

$$u = x \quad \xrightarrow{\text{differentiate}} \quad du = 1 \cdot dx$$

$$dv = \cos x \, dx \quad \xrightarrow{\text{integrate}} \quad v = \sin x$$

Then

$$\int \underbrace{x}_{u} \underbrace{\cos x \, dx}_{dv} = uv - \int v \, du$$

$$= x\sin x - \underbrace{\int \sin x\,dx}_{= -\cos x + C}$$

$$= \boxed{x\sin x + \cos x + C}$$

A better strategy for picking use is the LIATE method.

- **L**ogs
- **I**nverse trig
- **A**lgebraic (x^n, polynomials)
- **T**rig
- **E**xponential

Let u be the first function on this list that appears in your integral. Let everything else be dv.

<u>Ex:</u> Evaluate $\int x^2 \ln x \, dx$

<u>Solution:</u> Using IBP and the LIATE method,

$$u = \ln x \qquad v = x^3/3$$

$$du = \frac{1}{x}\,dx \qquad dv = x^2\,dx$$

$$\therefore \int x^2 \ln x \, dx = uv - \int v\,du$$

$$= (\ln x)\cdot \frac{x^3}{3} - \int \frac{x^3}{3}\cdot\frac{1}{x}\,dx$$

$$= \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2\,dx$$

$$= \boxed{\frac{x^3}{3}\ln x - \frac{x^3}{9} + C}$$

Ex: Evaluate $\int x^2 e^x\,dx$

Solution: Let $u = x^2$ $\quad v = e^x$

$du = 2x\,dx \quad dv = e^x\,dx$

$$\int x^2 e^x\,dx = x^2e^x - \int e^x \cdot 2x\,dx$$

$$= x^2e^x - 2\underbrace{\int xe^x\,dx}_{\text{IBP again!}}$$

$u = x \quad v = e^x$

$du = dx \quad dv = e^x\,dx$

$$= x^2e^x - 2\left[xe^x - \int e^x\,dx\right]$$

$$= \boxed{x^2e^x - 2xe^x + 2e^x + C}$$

Sometimes IBP can help even when the integrand doesn't look like a product!

Ex: Evaluate $\int \ln x \, dx$

Solution: Let $u = \ln x$ $\qquad v = x$

$du = \frac{1}{x} \, dx \qquad dv = 1 \, dx$

Then $\int \ln x \, dx = (\ln x) \cdot x - \int x \cdot \frac{1}{x} \, dx$

$$= x \ln x - \int 1 \, dx$$

$$= \boxed{x \ln x - x + C}$$

Ex: Evaluate $\int \arctan x \, dx$

Solution: Let $u = \arctan x$ $\qquad v = x$

$du = \frac{1}{1+x^2} \, dx \qquad dv = 1 \, dx$

Then

$$\int \arctan x \, dx = x \cdot \arctan x - \int \frac{x}{1+x^2} \, dx$$

u-substitution!
Let $u = 1+x^2$, so
$du = 2x \, dx \Rightarrow dx = \frac{du}{2x}$

$$= x \cdot \arctan x - \int \frac{\cancel{x}}{u} \left(\frac{du}{2\cancel{x}} \right)$$

$$= x\cdot \arctan x - \frac{1}{2}\int \frac{1}{u}\,du$$

$$= x\cdot \arctan x - \frac{1}{2}\ln|u| + C$$

$$= \boxed{x\cdot \arctan x - \frac{1}{2}\ln|1+x^2| + C}$$

Here's one more interesting type of IBP problem...

Ex: Evaluate $\int e^x \sin x \, dx$

Solution: Use IBP with $u = \sin x$ $\quad v = e^x$

$du = \cos x \, dx \quad dv = e^x dx$

$$\int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx$$

IBP again!

$u = \cos x \quad v = e^x$

$du = -\sin x dx \quad dv = e^x dx$

$$= e^x \sin x - \left[e^x \cos x - \int e^x(-\sin x)\, dx \right]$$

$$= e^x \sin x - e^x \cos x - \underbrace{\int e^x \sin x \, dx}$$

Aha! This is exactly the integral we started with!

If $I = \int e^x \sin x \, dx$, then we've just shown that

$$I = e^x \sin x - e^x \cos x - I,$$

hence $2I = e^x(\sin x - \cos x)$, and therefore

$$\boxed{I = \int e^x \sin x \, dx = \frac{e^x(\sin x - \cos x)}{2} + C}$$

Using integration by parts with definite integrals is no different — just don't forget the bounds!

$$\boxed{\int_a^b u \, dv = \Big[uv\Big]_a^b - \int_a^b v \, du}$$

Ex: Evaluate $\int_1^e x \ln x \, dx$.

Solution: Let $u = \ln x$ $\qquad v = \frac{x^2}{2}$

$du = \frac{1}{x} dx \qquad dv = x \, dx$

$$\int_1^e x \ln x \, dx = \left[\frac{x^2}{2} \ln x\right]_1^e - \int_1^e \frac{x^2}{2} \cdot \frac{1}{x} \, dx$$

$$= \left(\frac{e^2}{2}\underbrace{\ln e}_{=1} - \frac{1^2}{2}\underbrace{\ln 1}_{=0}\right) - \int_1^e \frac{x^2}{2}\,dx$$

$$= \frac{e^2}{2} - \left[\frac{x^2}{4}\right]_1^e$$

$$= \frac{e^2}{2} - \left(\frac{e^2}{4} - \frac{1^2}{4}\right) = \boxed{\frac{e^2}{4} + \frac{1}{4}}$$

Additional Exercises:

1. Evaluate $\int \sin x \cos x \, e^{\sin x} \, dx$

2. Evaluate $\int \ln(1+x^2)\,dx$

3. Let n be a positive integer with $n \geq 3$. Using IBP with $u = \sec^{n-2} x$ and $dv = \sec^2 x\,dx$, prove the reduction formula:

$$\int \sec^n x\,dx = \frac{1}{n-1}\sec^{n-2} x \tan x + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx$$

Solutions:

(a) Let's first tidy up the integral using a u-sub.

$$\int \sin x \cos x \, e^{\sin x} \, dx = \int \underbrace{\sin x \cdot e^{\sin x}}_{= ue^u} \cdot \underbrace{\cos x \, dx}_{= du}$$

Let $u = \sin x$, so

$du = \cos x \, dx$

$$= \int ue^u \, du \xrightarrow{\text{IBP}} = wv - \int v \, dw \quad \text{where}$$

$w = u$ $\qquad$ $v = e^u$

$dw = du$ $\qquad$ $dv = e^u du$

$$= ue^u - \int e^u \, du$$

$$= ue^u - e^u + C$$

$$= \boxed{\sin x \, e^{\sin x} - e^{\sin x} + C}$$

(b) Using IBP with

$u = \ln(1+x^2)$ $\qquad$ $v = x$

$du = \dfrac{2x}{1+x^2} dx$ $\qquad$ $dv = 1 \, dx$

we have

$$\int \ln(1+x^2) \, dx = x \cdot \ln(1+x^2) - \int \frac{2x^2}{1+x^2} \, dx$$

Let $x = \tan\theta$

$dx = \sec^2\theta \, d\theta$

$$= x \cdot \ln(1+x^2) - \int \frac{2\tan^2\theta}{\underbrace{1+\tan^2\theta}_{=\sec^2\theta}} \cdot \sec^2\theta \, d\theta$$

$$= x \cdot \ln(1+x^2) - 2\int \tan^2\theta \, d\theta$$

$$= x \cdot \ln(1+x^2) - 2\int (\sec^2\theta - 1) \, d\theta$$

$$= x \cdot \ln(1+x^2) - 2(\tan\theta - \theta) + C$$

$$= \boxed{x \cdot \ln(1+x^2) - 2x + 2\arctan(x) + C}$$

[Alternatively, $\int \frac{2x^2}{1+x^2} dx$ can be evaluated by writing

$\frac{2x^2}{1+x^2} = 2 - \frac{2}{1+x^2}$ using polynomial long division.

Then $\int \frac{2x^2}{1+x^2} dx = \int \left(2 - \frac{2}{1+x^2}\right) dx = \underline{2x - 2\arctan x + C}$.]

(c) We have

$u = \sec^{n-2} x$	$v = \tan x$
$du = (n-2)\sec^{n-3} x \cdot \sec x \tan x \, dx$ $= (n-2)\sec^{n-2} x \tan x \, dx$	$dv = \sec^2 x \, dx$

Hence,

$$\int \sec^n x \, dx = \sec^{n-2} x \tan x - \int (n-2) \sec^{n-2} x \tan^2 x \, dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x \left(\sec^2 x - 1\right) dx$$

$$= \sec^{n-2} x \tan x - (n-2) \underbrace{\int \sec^n x \, dx}_{\text{This is a multiple of the original integral! Move to LHS!}} + (n-2) \int \sec^{n-2} x \, dx$$

$$\Rightarrow (n-1) \int \sec^n x \, dx = \sec^{n-2} x \tan x + (n-2) \int \sec^{n-2} x \, dx$$

$$\xRightarrow{\div (n-1)} \int \sec^n x \, dx = \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx \quad \blacksquare$$