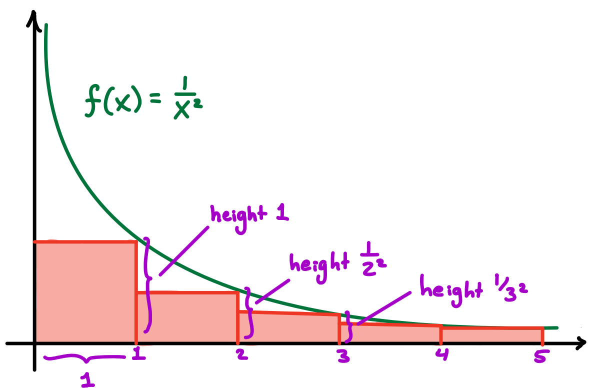
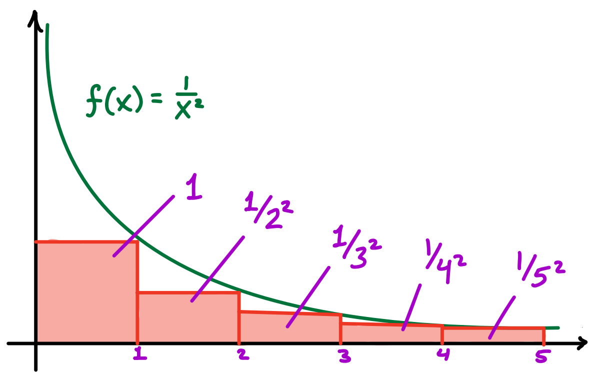
§5.6 – The Integral Test

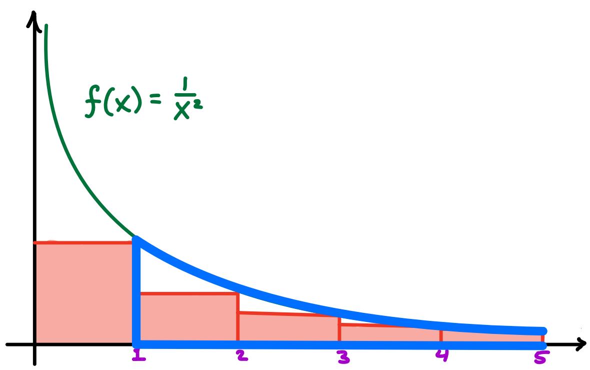
We'll introduce this test by studying the convergence of $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$

> **Idea:** Think of each term $\frac{1}{n^2}$ like the area of a $\frac{1}{n^2} \times 1$ rectangle.





Thus, the shaded area is exactly $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Is this area finite? Well... if the area under $f(x) = \frac{1}{x^2}$ from 1 to ∞ is finite, then the shaded area must also be finite!



Note that

$$\text{Area under } f(x) = \int_1^{\infty} \frac{1}{x^2}\, dx$$

which is convergent (i.e., finite) by the p-test.

Thus, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ must also be convergent! (i.e., finite)

The above argument is the main idea behind...

The Integral Test

Suppose $f(x)$ is continuous, positive, and decreasing for $x \in [1, \infty)$, and let $a_n = f(n)$, $n \in \mathbb{N}$.

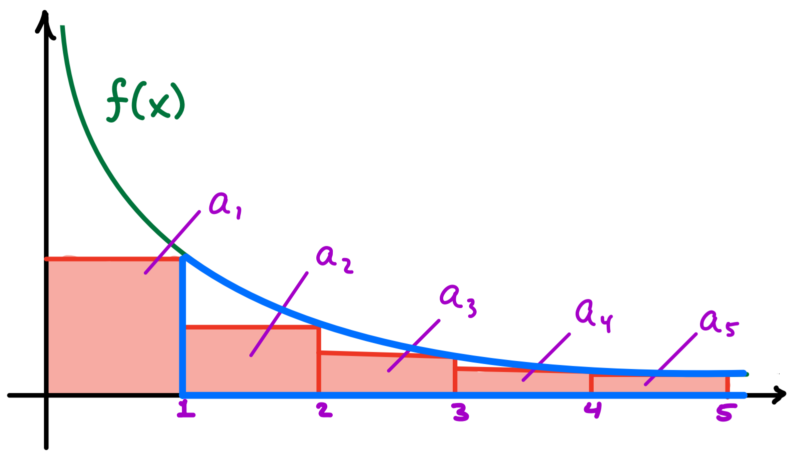
(i) If $\int_1^{\infty} f(x)\, dx$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

(ii) If $\int_1^{\infty} f(x)\, dx$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Idea of the proof:

Think of each a_n as the area of a rectangle with width 1 and height $f(n) = a_n$.

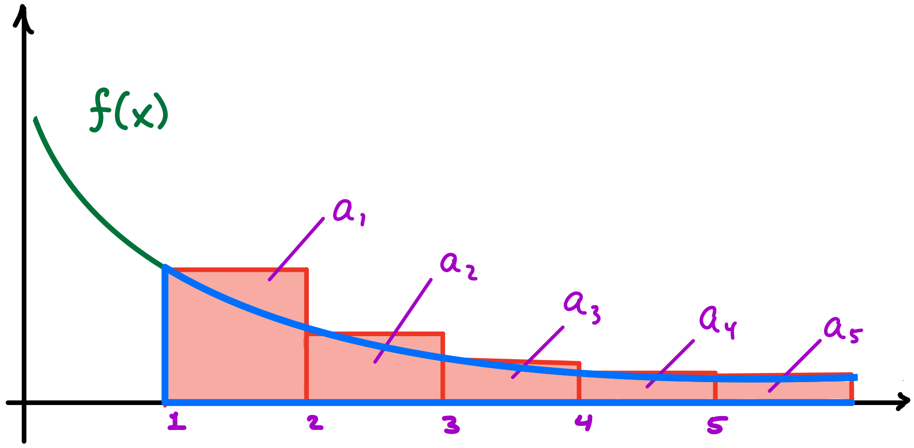
When drawn using right endpoints for the height, the rectangles lie under the graph of f, since f is decreasing.



We see that $\sum_{n=1}^{\infty} a_n \le a_1 + \int_1^{\infty} f(x)\,dx$. Consequently, if $\int_1^{\infty} f(x)\,dx$ converges, then $\sum_{n=1}^{\infty} a_n$ must also converge.

This is statement (i).

If instead we draw the rectangles with left endpoints for height...



... we see that the rectangles now lie above the graph (again, since f is decreasing), so $\int_1^\infty f(x)\,dx \le \sum_{n=1}^\infty a_n$

Consequently, if $\int_1^\infty f(x)\,dx$ diverges, $\sum_{n=1}^\infty a_n$ must diverge too. This is statement (ii) ■

Remarks:

1. Always verify the assumptions when using the integral test.

2. It is enough for the assumptions to hold on an interval $[K,\infty)$ where K is a positive real number. In this case,

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \int_K^{\infty} f(x)\,dx \text{ converges.}$$

3. The integral test may be helpful if the terms of the series look like they can be easily integrated.

Ex: Does $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(1+\ln n)}$ converge or diverge?

Solution: Let $f(x) = \dfrac{1}{x(1+\ln x)}$. Clearly, for $x \in [1,\infty)$, f is continuous, positive, and decreasing (since the numerator is constant and the denominator is increasing).

Hence, the integral test applies. We have

$$\int_1^\infty \frac{1}{x(1+\ln x)}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x(1+\ln x)}\,dx$$

$$= \lim_{t\to\infty}\int_1^{\ln t} \frac{1}{u}\,du \quad \left(u = 1+\ln x,\ du = \frac{1}{x}dx\right)$$

$$= \lim_{t\to\infty} \ln|\ln t| - \ln|1| = \infty$$

Since $\int_1^\infty \frac{1}{x(1+\ln x)}\,dx$ diverges, $\sum_{n=1}^\infty \frac{1}{n(1+\ln n)}$ diverges too, by the integral test.

Ex: Does $\sum_{n=1}^\infty \frac{n}{n^2+6}$ converge or diverge?

Solution: Let's try the divergence test!

$$\lim_{n\to\infty} \frac{n}{n^2+6} \overset{LH}{=} \lim_{n\to\infty}\frac{1}{2n} = 0 \quad \Rightarrow \quad \text{No conclusion!}$$

Hmm... okay, let's try the integral test!

Let $f(x) = \frac{x}{x^2+6}$. Note that f is continuous (since f is a rational function and $x^2+6 \neq 0$) and positive for $x>0$.

Is f decreasing? Not clear — let's compute $f'(x)$!

$$f(x) = \frac{x}{x^2+6} \quad \Rightarrow \quad f'(x) = \frac{(x^2+6)\cdot 1 - x\cdot(2x)}{(x^2+6)^2} = \frac{6-x^2}{(x^2+6)^2}$$

quotient rule!

$$\Rightarrow \quad f'(x) < 0 \quad \text{when} \quad 6-x^2 < 0$$

$$\Rightarrow \quad f'(x) < 0 \quad \text{when} \quad x > \sqrt{6}$$

$$\Rightarrow \quad f \text{ is decreasing for } x > \sqrt{6} \approx 2.45$$

$\therefore$ The assumptions of the integral test are satisfied on $[\sqrt{6}, \infty)$. We have

$$\int_{\sqrt{6}}^{\infty} \frac{x}{x^2+6}\,dx = \infty \quad \text{(exercise!)}$$

and therefore $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+6}$ diverges by the integral test.

Q: For which values of p does the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converge? (We call this a p-series)

A: If $p<0$, then $\lim_{n\to\infty} \frac{1}{n^p} = \infty$, and if $p=0$, then $\frac{1}{n^p} = \frac{1}{n^0} = 1$ for all n. In either case, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges by the divergence test.

If instead $p>0$, then $f(x) = \frac{1}{x^p}$ is continuous, positive, and decreasing for $x \in [1,\infty)$. By the integral test, since

$$\int_1^{\infty} \frac{1}{x^p}\,dx \text{ converges } \iff p>1 \quad \text{(by the p-test)}$$

it follows that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges $\iff$ $p>1$.

We have just proven...

The p-Series Test

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges when $p>1$ and diverges when $p \le 1$.

Ex: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3} = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots$

This is a convergent p-series, since $p = 3 > 1$.

Ex: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots$

This is a divergent p-series, since $p = 1/2 \leq 1$.

Application: Estimating Sums

We've seen that finding the exact sum of a convergent series $S = \displaystyle\sum_{n=1}^{\infty} a_n$ is often VERY hard. However, we can always approximate S using a partial sum:

$$\boxed{S \approx S_N = \sum_{n=1}^{N} a_n .}$$

The error (or remainder) in this approximation is denoted by R_N and is given by

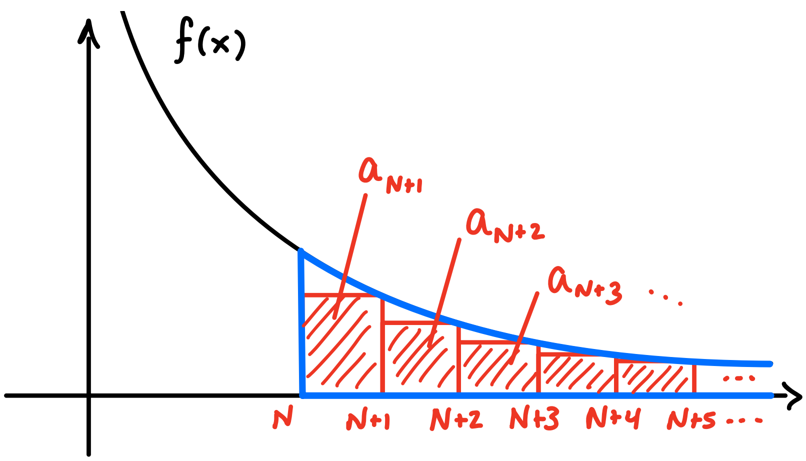
$$\text{Error} = R_N \overset{\text{Def.}}{=} S - S_N$$

$$= \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n$$

How big/small is this error?

$$= a_{N+1} + a_{N+2} + a_{N+3} + \cdots$$

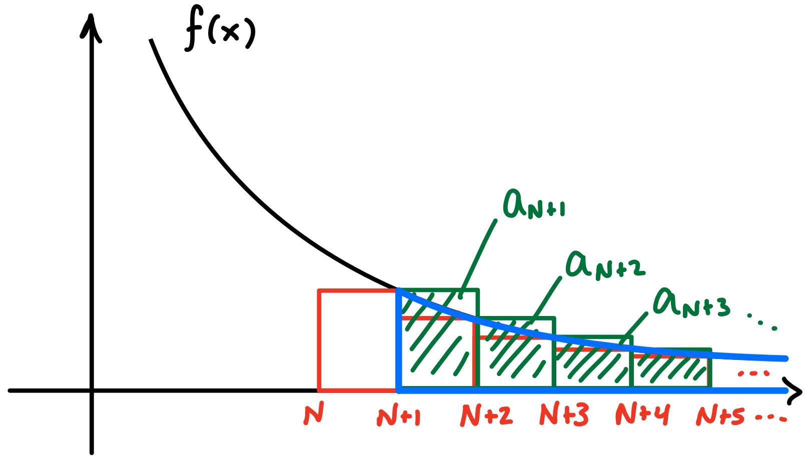
If $\sum_{n=1}^{\infty} a_n$ converges by the integral test, we are able to estimate the error! Indeed, suppose $a_n = f(n)$, where f is continuous, positive, and decreasing on $[1, \infty)$.



From the picture, we see that...

$$\boxed{R_N = a_{N+1} + a_{N+2} + a_{N+3} + \cdots \leq \int_N^{\infty} f(x)\,dx}$$

Let's now view the picture slightly differently, using left endpoints for the rectangles instead of right.



We now see that

$$\boxed{R_N = a_{N+1} + a_{N+2} + a_{N+3} + \cdots \geq \int_{N+1}^{\infty} f(x)\,dx}$$

Combining both of these estimates, we have ...

The Integral Test Estimation Theorem

If $\sum_{n=1}^{\infty} a_n$ converges by the integral test (so, in particular, $a_n = f(n)$ where $f(x)$ is continuous, positive, and decreasing for $x \in [1, \infty)$), then the error in approximating the sum S by S_N satisfies

$$\boxed{\int_{N+1}^{\infty} f(x)\,dx \leq R_N \leq \int_{N}^{\infty} f(x)\,dx.}$$

Ex: The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the integral test (check as an exercise!) If we use S_{10} to approximate the sum, S, we get

$$S \approx S_{10} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{10^2} \approx 1.5498.$$

(a) Estimate the error in this approximation

Solution: Let $f(x) = \frac{1}{x^2}$. We have

$$\int_{10+1}^{\infty} \frac{1}{x^2}\,dx \leq R_{10} \leq \int_{10}^{\infty} \frac{1}{x^2}\,dx$$

$$\Rightarrow \lim_{t\to\infty} \left[\frac{-1}{x}\right]_{11}^{t} \le R_{10} \le \lim_{t\to\infty} \left[\frac{-1}{x}\right]_{10}^{t}$$

$$\Rightarrow \lim_{t\to\infty} \left(\cancelto{0}{\frac{-1}{t}} + \frac{1}{11}\right) \le R_{10} \le \lim_{t\to\infty} \left(\cancelto{0}{\frac{-1}{t}} + \frac{1}{10}\right)$$

$$\Rightarrow \boxed{\underbrace{\frac{1}{11}}_{\approx 0.09} \le R_{10} \le \underbrace{\frac{1}{10}}_{= 0.1}}$$

(b) How large or small could S possibly be?

<u>Solution:</u> From (a), we have

$$0.09 \le R_{10} \le 0.1$$

$$\Rightarrow 0.09 \le S - S_{10} \le 0.1$$

$$\Rightarrow \underset{+1.5498}{0.09} \le \underset{+1.5498}{S - 1.5498} \le \underset{+1.5498}{0.1}$$

$$\Rightarrow \boxed{1.6398 \le S \le 1.6498}$$

$$\left(\text{The actual sum is } S = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.645\right)$$

(c) How many terms N are needed in order to guarantee an error R_N less than $\frac{1}{1000}$?

Solution: We know that $R_N \leq \int_N^\infty \frac{1}{x^2}\,dx = \frac{1}{N}$

and so we want $\frac{1}{N} < \frac{1}{1000}$, or $N > 1000$.

Thus, $\boxed{N = 1001}$ will suffice.

Additional Exercises:

Ex: Show that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\,(\sqrt{n}+1)^2}$ converges by the integral test, then find an upper bound on the error when using S_{100} to approximate the sum, S.

Solution: Let $f(x) = \dfrac{1}{\sqrt{x}(\sqrt{x}+1)^2}$. We note that f is continuous, positive, and decreasing on $[1,\infty)$, hence the integral test applies. We have

$$\int_1^\infty f(x)\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{\sqrt{x}(\sqrt{x}+1)^2}\,dx \qquad u = \sqrt{x}+1 \quad du = \frac{1}{2\sqrt{x}}\,dx$$

$$= \lim_{t\to\infty} \int_2^{\sqrt{t}+1} \frac{2}{u^2}\,du$$

$$= \lim_{t\to\infty} \left[\frac{-2}{u} \right]_2^{\sqrt{t}+1}$$

$$= \lim_{t\to\infty} \left(\frac{-2}{\sqrt{t}+1} + \frac{2}{2} \right) = 1$$

Since $\int_1^\infty f(x)\,dx$ converges, so too does $\sum_{n=1}^\infty \frac{1}{\sqrt{n}(\sqrt{n}+1)^2}$.

For the error, note that

$$R_{100} \leq \int_{100}^\infty \frac{1}{\sqrt{x}(\sqrt{x}+1)^2}\,dx \qquad u = \sqrt{x}+1 \quad du = \frac{1}{2\sqrt{x}}\,dx$$

$$= \lim_{t \to \infty} \int_{\sqrt{100}+1}^{\sqrt{t}+1} \frac{2}{u^2}\, du$$

$$= \lim_{t \to \infty} \left[\frac{-2}{u} \right]_{11}^{\sqrt{t}+1}$$

$$= \lim_{t \to \infty} \left(\frac{-2}{\sqrt{t}+1} + \frac{2}{11} \right)$$

$$= \boxed{\frac{2}{11} = 0.\overline{18}}$$