§2.4 – Improper Integrals

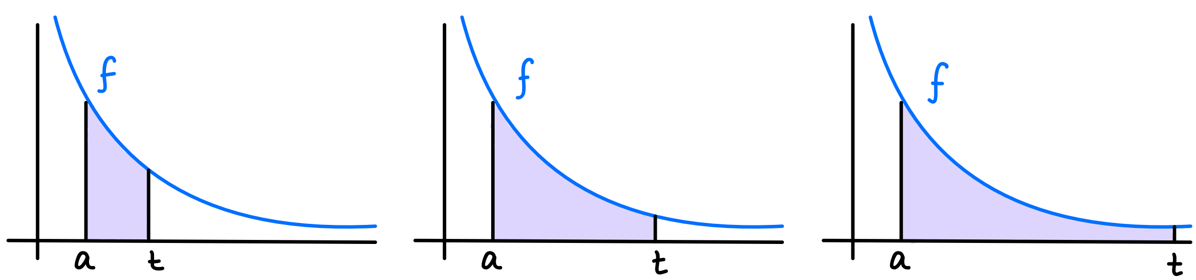
In this section we'll learn how to handle integrals over infinite domains and integrals of functions with an infinite discontinuity (i.e., a vertical asymptote).

Integrals of these types are known as improper integrals.

Type I : Infinite Domains

Definition [Type I Improper Integral]: We define

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$$



and similarly,

$$\int_{-\infty}^{a} f(x)\,dx = \lim_{t \to -\infty} \int_{t}^{a} f(x)\,dx$$

The integral <u>converges</u> if the limit exists. It <u>diverges</u> if the limit DNE (i.e., doesn't approach anything or is $\pm\infty$). We also define

(Deal with each infinity separately!)

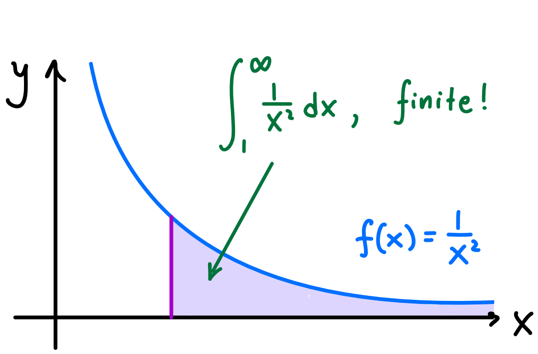
$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{t \to -\infty} \int_{t}^{0} f(x)\,dx + \lim_{s \to \infty} \int_{0}^{s} f(x)\,dx$$

If both limits exist, we say $\int_{-\infty}^{\infty} f(x)\,dx$ <u>converges</u>.

If even one limit DNE, $\int_{-\infty}^{\infty} f(x)\,dx$ <u>diverges</u>.

<u>Examples:</u>

(a) $$\int_{1}^{\infty} \frac{1}{x^2}\,dx = \lim_{t \to \infty} \int_{1}^{t} \frac{1}{x^2}\,dx = \lim_{t \to \infty} \left[\frac{-1}{x}\right]_{1}^{t}$$



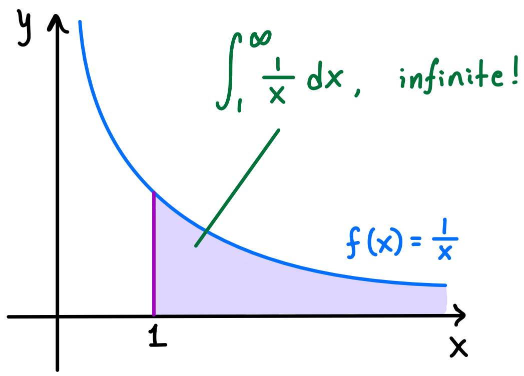
$$= \lim_{t\to\infty} \left[\underbrace{\frac{-1}{t}}_{\to 0} - \left(\frac{-1}{1}\right) \right]$$

$$= 1$$

$\therefore$ Integral converges!

So the area is finite (and equal to 1!)

(b) $$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty} \Big[\ln|x| \Big]_1^t$$



$$= \lim_{t\to\infty} \underbrace{\ln|t|}_{\to\infty} - \ln 1$$

$$= \infty$$

$\therefore$ Integral diverges!

$$\boxed{\text{For which values of } p \text{ does } \int_1^\infty \frac{1}{x^p}\,dx \text{ converge?}}$$

Well... From (a), $\int_1^{\infty} \frac{1}{x^p}\,dx$ diverges when $p=1$.

For $p \neq 1$, we have

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{t\to\infty} \int_1^t x^{-p}\,dx = \lim_{t\to\infty} \left[\frac{x^{-p+1}}{-p+1}\right]_1^t$$

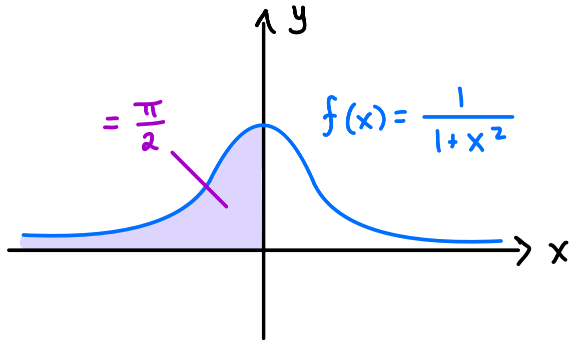
$$= \lim_{t\to\infty} \left(\frac{t^{-p+1}}{-p+1} - \frac{1}{-p+1}\right)$$

Note that $t^{-p+1} \longrightarrow \infty$ if $-p+1>0$ (i.e., $p<1$), in which case the integral diverges. But if $-p+1<0$ (i.e., $p>1$), then $t^{-p+1} \longrightarrow 0$ and the integral converges. In summary...

Theorem [Convergence of p-Integrals]

$\int_1^{\infty} \frac{1}{x^p}\,dx$ converges for $p>1$ and diverges for $p \le 1$.

(c) $\displaystyle\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx = \lim_{t\to-\infty} \int_{t}^{0} \frac{1}{1+x^2}\,dx = \lim_{t\to-\infty} \Big[\arctan x\Big]_{t}^{0}$



$$= \lim_{t\to-\infty} \Big[\underbrace{\arctan(0)}_{=0} - \underbrace{\arctan(t)}_{\to -\pi/2}\Big]$$

$$= \boxed{\pi/2}$$

Thus, the integral converges!

(d) $\displaystyle\int_{-\infty}^{\infty} x\cdot\cos(x^2)\,dx = \lim_{t\to-\infty} \int_{t}^{0} x\cos(x^2)\,dx + \underbrace{\lim_{s\to\infty} \int_{0}^{s} x\cos(x^2)\,dx}_{\text{Let's try computing this first.}}$

$$\lim_{s\to\infty} \int_{0}^{s} x\cos(x^2)\,dx = \lim_{s\to\infty} \frac{1}{2}\int_{1}^{s^2} \cos(u)\,du$$

$u = x^2$
$du = 2x\,dx$

$$= \lim_{s\to\infty} \frac{1}{2}\Big[\sin(u)\Big]_{0}^{s^2}$$

$$= \lim_{s\to\infty} \frac{1}{2}\sin(s^2)$$

Oscillates, doesn't approach anything

$$\Rightarrow \lim_{s\to\infty} \int_0^s x\cos(x^2)\,dx \ \text{ DNE}$$

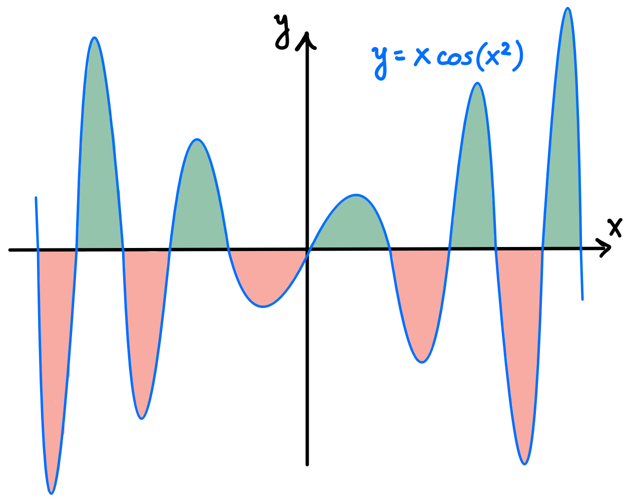
$$\Rightarrow \int_{-\infty}^{\infty} x\cos(x^2)\,dx \ \ \underline{\text{diverges.}}$$

Recall: If even one of the limits DNE, the integral diverges. In this case, there is no need to check the other limit!

Note: It is not correct to compute $\int_{-\infty}^{\infty} f(x)\,dx$ as $\lim_{t\to\infty}\int_{-t}^{t} f(x)\,dx$.

This is called Cauchy Principal Value and is different from our definition!

In this example, you would get $\lim_{t\to\infty}\int_{-t}^{t} x\cdot\cos(x^2)\,dx = 0\ldots$



... but this doesn't agree with our definition of convergence, which requires both limits to exist!

Properties of Type I Improper Integrals

Suppose that $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ converge.

1. $\int_a^\infty [\alpha f(x) + \beta g(x)]\,dx$ converges for all $\alpha, \beta \in \mathbb{R}$ and

$$\alpha \int_a^\infty f(x)\,dx + \beta \int_a^\infty g(x)\,dx$$

2. If $f(x) \leq g(x)$ for all $x \geq a$, then

$$\int_a^\infty f(x)\,dx \leq \int_a^\infty g(x)\,dx$$

3. If $a < c < \infty$, then $\int_c^\infty f(x)dx$ converges and

$$\int_a^\infty f(x)\,dx = \int_a^c f(x)\,dx + \int_c^\infty f(x)\,dx$$

Sometimes we can determine whether a type I integral converges / diverges without computing it exactly!

The Comparison Theorem for Type I Integrals

Assume f and g are continuous on $[a,\infty)$ and

$0 \le f(x) \le g(x)$ for all $x \geq a$.

1. If $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges.

2. If $\int_a^\infty f(x)\,dx$ diverges, then $\int_a^\infty g(x)\,dx$ diverges.

Remarks:

(i) It is often useful to compare with $\int_1^\infty \frac{1}{x^p}\,dx$,

or perhaps $\int_0^\infty e^{-x}\,dx$ (which converges — show this!)

(ii) We cannot make any conclusions if $\int_a^\infty f(x)\,dx$

converges or if $\int_a^\infty g(x)dx$ diverges!

Examples: Do the following converge or diverge?

(a) $\displaystyle\int_1^\infty \frac{1}{x^5+1}$ (Computing this exactly is a **HARD** PFD problem!)

Solution: Note that on $[1,\infty)$,

$$0 \le \frac{1}{x^5+1} \le \frac{1}{x^5}.$$

Since $\displaystyle\int_1^\infty \frac{1}{x^5}\,dx$ is a convergent $\underbrace{\text{p-integral}}_{p=5\ (>1)}$, $\displaystyle\int_1^\infty \frac{1}{x^5+1}\,dx$ converges by comparison

(b) $\displaystyle\int_1^\infty \frac{1}{x+\sqrt{x}}\,dx$

Solution: Let's try to compare with something simpler!

Idea 1: $\displaystyle 0 \le \frac{1}{x+\sqrt{x}} \le \frac{1}{x}$ (But $\displaystyle\int_1^\infty \frac{1}{x}\,dx$ diverges $\Rightarrow$ No conclusions!)

Idea 2: $\displaystyle 0 \le \frac{1}{x+\sqrt{x}} \le \frac{1}{\sqrt{x}}$ (But $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ diverges $\Rightarrow$ No conclusions!)

Idea 3: $\dfrac{1}{x+\sqrt{x}} \geqslant \dfrac{1}{x+x} = \dfrac{1}{2x} \geqslant 0$ and

$\displaystyle\int_1^\infty \frac{1}{2x}\,dx$ diverges $\Rightarrow \displaystyle\int_1^\infty \frac{1}{x+\sqrt{x}}\,dx$ diverges by comparison.

(c) $\displaystyle\int_1^\infty \frac{e^{2x}}{x+e^{3x}}\,dx$

Solution: $0 \leq \dfrac{e^{2x}}{x+e^{3x}} \leq \dfrac{e^{2x}}{e^{3x}} = e^{-x}$ and $\displaystyle\int_1^\infty e^{-x}\,dx$

converges, so $\displaystyle\int_1^\infty \frac{e^{2x}}{x+e^{3x}}\,dx$ converges by comparison.

(d) $\displaystyle\int_0^\infty e^{-x^2}\,dx$

Solution: Note that $x \leq x^2$ for $x \geq 1$

$$\Rightarrow e^x \leq e^{x^2} \text{ for } x \geq 1$$

$$\Rightarrow \frac{1}{e^{x^2}} \leq \frac{1}{e^x} \text{ for } x \geq 1$$

$$\Rightarrow e^{-x^2} \leq e^{-x} \text{ for } x \geq 1.$$

Since $\int_1^\infty e^{-x}\,dx$ converges, so does $\int_1^\infty e^{-x^2}dx$ by comparison. Thus, $\int_0^\infty e^{-x^2}dx$ converges.

Note: $\int_0^1 e^{-x^2}dx$ isn't improper and won't affect convergence.

The comparison theorem is very helpful, but what can we do if $f(x)$ takes on a mix of positive and negative values? In this case, we can try the following!

Theorem

If $\int_a^\infty |f(x)|\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges.

Notes:

① If $\int_a^\infty |f(x)|\,dx$ converges, we say that $\int_a^\infty f(x)\,dx$

converges absolutely. The theorem says that absolute convergence implies convergence.

② The converse to this theorem is false, though proving this is tricky! It turns out that $\int_1^\infty \frac{\sin x}{x}\,dx$ converges but not absolutely $\left(\int_1^\infty \left|\frac{\sin x}{x}\right| dx \text{ diverges}\right)$.

Proof: If $\int_a^\infty |f(x)|\,dx$ converges, then so too does $\int_a^\infty 2|f(x)|\,dx$. Since $0 \le f(x) + |f(x)| \le 2|f(x)|$, by the comparison theorem, $\int_a^\infty f(x) + |f(x)|\,dx$ converges.

Consequently, $\int_a^\infty f(x)\,dx = \int_a^\infty f(x) + |f(x)|\,dx - \int_a^\infty |f(x)|\,dx$ converges. ■

Ex: Does $\displaystyle\int_1^\infty \frac{\sin x}{x^2+3}\,dx$ converge?

Can't use comparison directly since sometimes $\sin(x) < 0$ ...

Solution: Checking for absolute convergence will be easier!

$$0 \le \left|\frac{\sin x}{x^2+3}\right| = \frac{|\sin x|}{x^2+3} \le \frac{1}{x^2+3} \le \frac{1}{x^2}$$

By comparison, since $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ converges, so does $\displaystyle\int_1^\infty \left|\frac{\sin x}{x^2+3}\right| dx$.

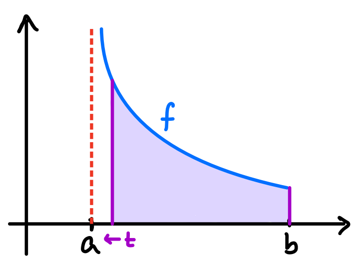
Thus, $\displaystyle\int_1^\infty \frac{\sin x}{x^2+3}\,dx$ converges absolutely, hence it converges.

Type II: Integrands with an Infinite Discontinuity

Definition [Type II Improper Integral]:

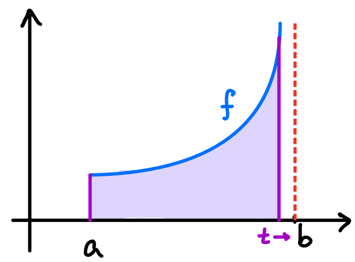
(i) If f has an infinite discontinuity at $x = a$, we define

$$\boxed{\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx}$$



(ii) If f has an infinite discontinuity at $x=b$, we define

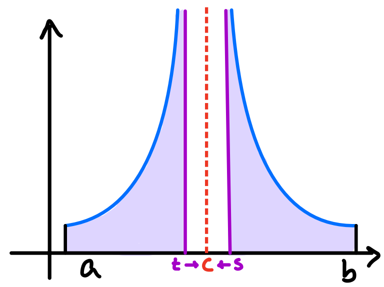
$$\boxed{\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx}$$



(iii) If has an infinite discontinuity at $x=c$ with $a<c<b$,

we define

$$\boxed{\int_a^b f(x)\,dx = \lim_{t\to c^-}\int_a^t f(x)\,dx + \lim_{s\to c^+}\int_s^b f(x)\,dx}$$



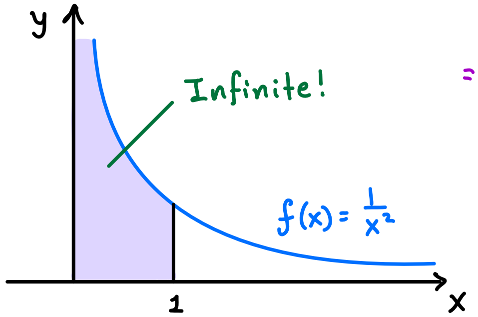
The integral converges if (all) its limit(s) exist.

If even one limit DNE, the integral diverges.

Examples:

(a) $\displaystyle\int_0^1 \frac{1}{x^2}\,dx = \lim_{t\to 0^+}\int_t^1 \frac{1}{x^2}\,dx$

problem at $x=0$



$$= \lim_{t \to 0^+} \left[\frac{1}{t} - \frac{1}{1} \right] = \infty$$

Thus, the integral <u>diverges.</u>

Exercise: Show that $\int_0^1 \frac{1}{x^p}\, dx$ converges for $p < 1$ and diverges for $p \geq 1$

(b) $\int_1^2 \frac{x}{x^2 - 4}\, dx$

problem at 2, but we'll start with a substitution.

Let $u = x^2 - 4$, $\quad x = 2 \Rightarrow u = 0$

$du = 2x\, dx \qquad x = 1 \Rightarrow u = -3$

$$= \int_{-3}^{0} \frac{\cancel{x}}{u} \cdot \frac{du}{2\cancel{x}}$$

problem at 0

$$= \lim_{t \to 0^-} \frac{1}{2} \int_{-3}^{t} \frac{1}{u}\, du$$

$$= \lim_{t \to 0^-} \frac{1}{2} \left[\ln|t| - \ln|-3| \right] = -\infty$$

$-\infty$

Limit DNE!

$\Rightarrow$ Integral <u>diverges!</u>

problem at $x = 1$.

(c) $$\int_0^4 \frac{1}{\sqrt[3]{x-1}}\,dx = \lim_{t\to 1^-}\int_0^t (x-1)^{-1/3}\,dx + \lim_{s\to 1^+}\int_s^4 (x-1)^{-1/3}\,dx$$

Let $u = x-1$

$du = dx$

$$= \lim_{t\to 1^-}\int_{-1}^{t-1} u^{-1/3}\,du + \lim_{s\to 1^+}\int_{s-1}^{3} u^{-1/3}\,du$$

$$= \lim_{t\to 1^-}\frac{3}{2}\left[\underbrace{(t-1)^{2/3}}_{\to 0} - \underbrace{(-1)^{2/3}}_{=1}\right] + \lim_{s\to 1^+}\frac{3}{2}\left[3^{2/3} - \underbrace{(s-1)^{2/3}}_{\to 0}\right]$$

$$= \frac{3}{2}\left[3^{2/3} - 1\right]$$ Finite!

$\therefore$ Integral converges.