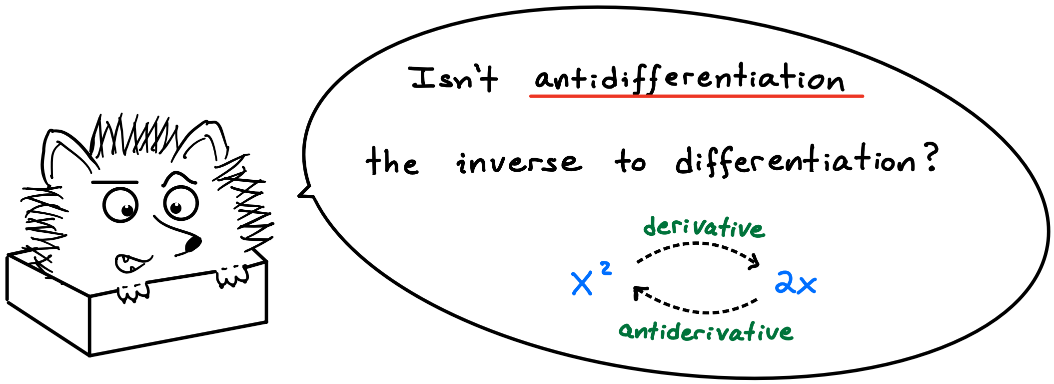
§1.6 – The Fundamental Theorem of Calculus, Part II

From FTC I, it seems like differentiation and integration are inverse operations — and they are!



As we will soon see, integrals and antiderivatives are very closely related! Before stating this connection in Part II of the FTC, let's review what we know about antiderivatives.

> **Definition:** A function $F(x)$ is an antiderivative of $f(x)$ if $F'(x) = f(x)$.

Ex: If $f(x) = x^3$, then $F(x) = \frac{x^4}{4}$ is an antiderivative since $F'(x) = x^3$. But it's not the only antiderivative:

$$\frac{x^4}{4} + 1, \quad \frac{x^4}{4} - \frac{1}{2}, \quad \frac{x^4}{4} + \pi, \quad \text{etc.}$$

are all antiderivatives of x^3 too!

As an application of the MVT, one can prove that all antiderivatives of f have this form.

The Antiderivative Theorem:

If $F(x)$ and $G(x)$ are both antiderivatives of $f(x)$, then $G(x) = F(x) + C$ for some $C \in \mathbb{R}$.

The collection of *all* antiderivatives of a function $f(x)$ is called the indefinite integral of f, written $\int f(x)\,dx$.

That is,

$$\boxed{\int f(x)\,dx = F(x) + C, \quad C \in \mathbb{R}}$$

where $F(x)$ is any antiderivative of $f(x)$.

Some Common Antiderivatives

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad \text{for all } n \in \mathbb{R},\ n \neq -1$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \sec x \tan x\,dx = \sec x + C$$

$$\int \frac{1}{1+x^2}\,dx = \arctan x + C \qquad \int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$$

$$\int e^x\,dx = e^x + C \qquad \int a^x\,dx = \frac{a^x}{\ln a} + C$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

Why $|x|$? Well..., $\frac{1}{x}$ is defined for $x>0$ and $x<0$, so its antiderivative should be too!

It turns out that knowing an antiderivative of $f(x)$ gives us a very efficient way to compute $\int_a^b f(x)\,dx$ — without Riemann sums!! This is part II of the FTC!

The Fundamental Theorem of Calculus (FTC), Part II

If f is continuous on $[a,b]$ and F is any antiderivative of f, then

$$\int_a^b f(x)\,dx = \underbrace{\Big[F(x)\Big]_a^b}_{\text{Notation!}} = F(b) - F(a)$$

Proof: By FTC I, $G(x) = \int_a^x f(t)\,dt$ is an antiderivative of $f(x)$, so $G(x) = F(x) + C$ for some $C \in \mathbb{R}$. Hence,

$$\begin{aligned}
\int_a^b f(x)\,dx &= \int_a^b f(x)\,dx + \int_a^a f(x)\,dx \\
&= G(b) - G(a) \\
&= \big[F(b) + C\big] - \big[F(a) + C\big] \\
&= F(b) - F(a)
\end{aligned}$$

$\blacksquare$

Examples:

Painfully computed using Riemann sums in §1.2

(a) $\displaystyle\int_0^2 (4x^3 - x)\,dx = \left[x^4 - \frac{x^2}{2}\right]_0^2$

$$= \left(2^4 - \frac{2^2}{2}\right) - \left(\cancel{0^4} - \cancel{\frac{0^2}{2}}\right)$$

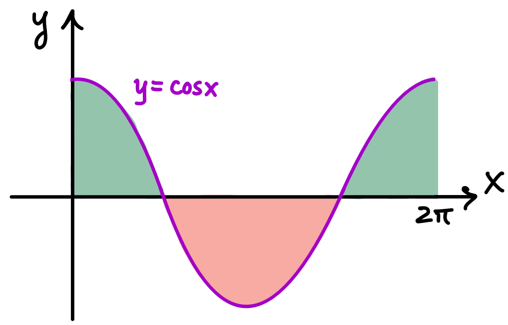
$$= 16 - 2$$

$$= \boxed{14}$$

(b) $\displaystyle\int_0^{2\pi} \cos x \, dx = \left[\sin x\right]_0^{2\pi}$

$$= \sin 2\pi - \sin 0$$

$$= \boxed{0}$$



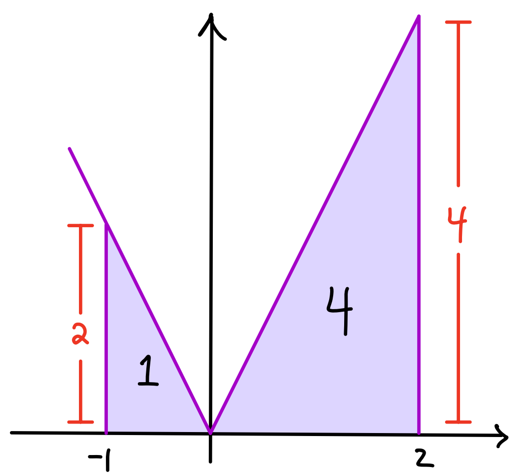
Makes sense geometrically. Signed area = 0.

(c) $\displaystyle\int_{-1}^{2} |2x| \, dx = \int_{-1}^{0} |2x| \, dx + \int_0^2 |2x| \, dx$

$$= \int_{-1}^{0} -2x \, dx + \int_0^2 2x \, dx$$

$$= -\left[x^2\right]_{-1}^{0} + \left[x^2\right]_0^2$$

$$= \boxed{5}$$



Also makes sense geometrically!

Knowing an antiderivative for $f(x)$ makes it easy to compute $\int_a^b f(x)\,dx$, but finding an antiderivate can be tricky! Sometimes, the integrand must first be manipulated.

Examples:

(a) $$\int \frac{(x+1)^2}{x}\,dx = \int \frac{x^2+2x+1}{x}\,dx$$
$$= \int \left(x+2+\frac{1}{x}\right)dx$$
$$= \boxed{\frac{x^2}{2} + 2x + \ln|x| + C}$$

(b) $$\int \left(1+\frac{1}{x}\right)\sqrt{x}\,dx = \int \left(\sqrt{x} + \frac{\sqrt{x}}{x}\right)dx$$
$$= \int \left(x^{1/2} + x^{-1/2}\right)dx$$
$$= \frac{x^{3/2}}{3/2} + \frac{x^{1/2}}{1/2} + C$$
$$= \boxed{\frac{2}{3}x^{3/2} + 2x^{1/2} + C}$$

(c) $\displaystyle\int \frac{\tan\theta}{\sin 2\theta}\,d\theta = \int \frac{\frac{\cancel{\sin\theta}}{\cos\theta}}{2\cancel{\sin\theta}\cos\theta}\,d\theta$

$$= \frac{1}{2}\int \frac{1}{\cos^2\theta}\,d\theta$$

$$= \frac{1}{2}\int \sec^2\theta\,d\theta = \boxed{\frac{1}{2}\tan\theta + C}$$

(d) $\displaystyle\int \frac{x^2}{x^2+1}\,dx = \int \frac{(x^2+1)-1}{x^2+1}\,dx$

$$= \int \left(1 - \frac{1}{x^2+1}\right)dx$$

Could have also used polynomial long division!

$$= \boxed{x - \arctan x + C}$$

<u>Our goal over the next several lessons</u>: Learn methods for computing antiderivatives for a variety of functions!