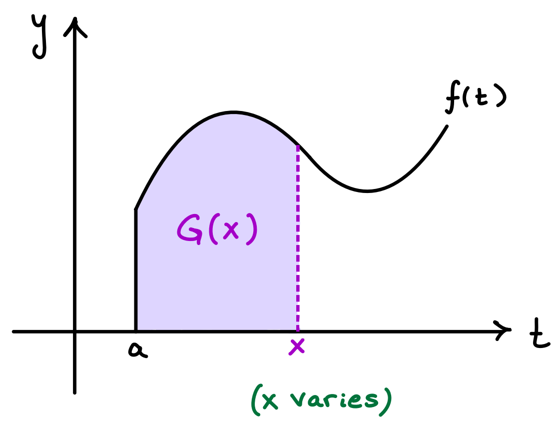
§1.5 – The Fundamental Theorem of Calculus, Part I

The FTC is a derivative rule that establishes a vital link between the worlds of differential and integral calculus. Its importance cannot be overstated.

To discuss the FTC, we must first explore the idea of an integral function. Specifically, if f is continuous on $[a,b]$, consider the function

$$G(x) = \int_a^x f(t)\,dt$$

Given x, $G(x)$ outputs the area under the graph of f from $t=a$ to $t=x$.



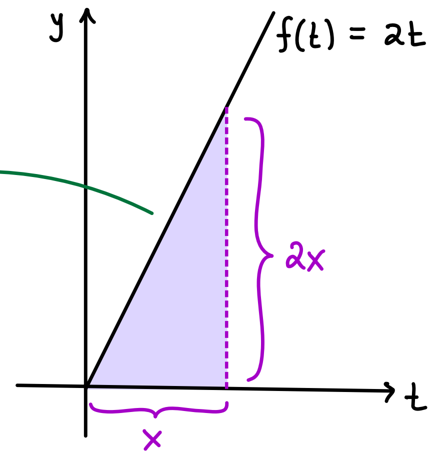
Example: Consider $f(x) = 2x$ for $x \in [0,4]$.

$$G(x) = \int_0^x 2t\,dt = \text{Area under } f(t) = 2t \text{ from } t=0 \text{ to } t=x$$

$$= \frac{1}{2}(\text{Base})(\text{Height})$$

$$= \frac{1}{2}(x)(2x)$$

$$= x^2$$



Notice: $G'(x) = 2x = f(x)$!

This connection turns out to be true generally — it's the first part of the FTC!

The Fundamental Theorem of Calculus (FTC) Part I

If f is continuous on an open interval I containing $x=a$, then the function

$$G(x) = \int_a^x f(t)\,dt$$

is differentiable for all $x \in I$ and $G'(x) = f(x)$. That is,

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

Proof: Given $x \in I$, we have

$$G'(x) = \lim_{h \to 0} \frac{G(x+h) - G(x)}{h}$$

$$= \lim_{h \to 0} \frac{\int_a^{x+h} f(t)\,dt - \int_a^{x} f(t)\,dt}{h}$$

$$= \underbrace{\lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)\,dt}_{\text{Average of } f(t) \text{ on } [x,\, x+h].}$$

Since f is continuous, by the Average Value Theorem, for each $h \neq 0$, there exists $c_h \in [x, x+h]$ such that

$$f(c_h) = \frac{1}{h} \int_x^{x+h} f(t)\,dt.$$

Since $x \le c_h \le x+h$, we have

$$\lim_{h \to 0} c_h = x,$$

by the squeeze theorem. Hence,

$$G'(x) = \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)\,dt = \lim_{h \to 0} f(c_h) \overset{\text{Since } f \text{ is continuous}}{=} f(x) \qquad \blacksquare$$

Example: What is $\dfrac{d}{dx}\displaystyle\int_1^x \cos(t^2)\,dt$?

Solution: Since $f(t) = \cos(t^2)$ is continuous, by FTC I,

$$\frac{d}{dx}\int_1^x \cos(t^2)\,dt = \boxed{\cos(x^2).}$$

Remarks

1. FTC I allows us to differentiate an integral function without having to first evaluate the integral!

2. FTC I tells us that differentiation "undoes" the integral — they are inverse operations! We'll explore this idea further when we discuss FTC II.

Extensions of FTC I

A function, $h(x)$!

Ex: What is $\dfrac{d}{dx}\displaystyle\int_0^{8x+1} \sqrt{1+t^3}\,dt$?

It turns out that when f is continuous, and h is differentiable,

$$\frac{d}{dx}\int_a^{h(x)} f(t)\,dt = f(h(x))\cdot h'(x)$$

Why? Well... if $G(x) = \int_a^x f(t)\,dt$, then $G'(x) = f(x)$ by FTC I, hence

$$\frac{d}{dx}\int_a^{h(x)} f(t)\,dt = \frac{d}{dx} G(h(x))$$

$$= G'(h(x))\cdot h'(x)$$

$$= f(h(x))\cdot h'(x),$$

as claimed. Now let's revisit our example!

Solution: Since $f(t) = \sqrt{1+t^3}$ is continuous,

$$\frac{d}{dx}\int_0^{8x+1}\sqrt{1+t^3}\,dt = \sqrt{1+(8x+1)^3}\cdot(8x+1)' = \boxed{\sqrt{1+(8x+1)^3}\cdot 8}$$

Ex: What is $\displaystyle\frac{d}{dx}\int_{2x}^{\sin x} e^{t^2}\,dt$?

Another function, $h(x)$

A function $g(x)$

Let's see...

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = \frac{d}{dx}\left(\int_{g(x)}^{a} f(t)\,dt + \int_{a}^{h(x)} f(t)\,dt\right)$$

$$= \frac{d}{dx}\left(\int_{a}^{h(x)} f(t)\,dt - \int_{a}^{g(x)} f(t)\,dt\right)$$

$$= f(h(x))\cdot h'(x) - f(g(x))\cdot g'(x)$$

This is the most general version of FTC I!

Extended FTC I: If f is continuous and g and h are differentiable, then

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = f(h(x))\cdot h'(x) - f(g(x))\cdot g'(x)$$

Solution: Since e^{t^2} is continuous, by the extended version of FTC I:

$$\frac{d}{dx}\int_{2x}^{\sin x} e^{t^2}\,dt = e^{(\sin x)^2}\cdot(\sin x)' - e^{(2x)^2}\cdot(2x)'$$

$$= \boxed{e^{\sin^2 x}\cdot\cos x - e^{4x^2}\cdot 2}$$