Series Convergence Tests

For series that are neither geometric nor telescoping, it can be VERY hard to find a nice expression for the partial sums, S_N. As a result, it is often VERY hard to find the exact sum of such a series!

e.g. We will soon be able to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$$

converges. But what's the sum?

$$S_2 = 1.25, \quad S_3 \approx 1.361, \quad S_4 \approx 1.424$$

Perhaps the sum is 1.5? 2? Nope! In 1735, after many prominent mathematicians failed to find the sum, Euler proved that

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.}$$

Proof: Beyond the scope of MATH 138!

From this point onward, we won't be interested in finding exact sums, but deciding whether a series converges or diverges. We have many tests for this!

§5.3 – The Divergence Test

Our first test is based on the following observation:

If $\sum_{n=1}^{\infty} a_n$ has any hope of converging, the terms a_n must become small (i.e., $a_n \longrightarrow 0$).

Thus, we get the following:

The Divergence Test

If $\lim_{n\to\infty} a_n \neq 0$ (or if $\lim_{n\to\infty} a_n$ DNE) then $\sum_{n=1}^{\infty} a_n$ diverges.

Proof: We will prove the contrapositive:

$$\boxed{\text{" If } \sum_{n=1}^{\infty} a_n \text{ converges, then } \lim_{n\to\infty} a_n = 0 \text{ "}}$$

So, suppose $\sum_{n=1}^{\infty} a_n$ converges. This means that

$\lim_{n\to\infty} S_n = S$ for some $S \in \mathbb{R}$, where $S_n = a_1 + a_2 + \cdots + a_n$

is the n^{th} partial sum. Thus,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \Big(\underbrace{(a_1 + a_2 + \cdots + a_n)}_{S_n} - \underbrace{(a_1 + a_2 + \cdots + a_{n-1})}_{S_{n-1}} \Big)$$

$$= \lim_{n\to\infty} \big(\underbrace{S_n}_{\to S} - \underbrace{S_{n-1}}_{\to S} \big)$$

$$= S - S$$

$$= 0.$$

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Ex: $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots$

$$\lim_{n\to\infty} \frac{n}{n+1} \overset{LH}{=} \lim_{n\to\infty} \frac{1}{1+0} = 1$$

Since $\lim\limits_{n\to\infty} \frac{n}{n+1} \neq 0$, $\sum\limits_{n=1}^{\infty} \frac{n}{n+1}$ <u>diverges</u> by the divergence test.

<u>Ex:</u> $\sum\limits_{n=1}^{\infty} \sec\left(\frac{1}{n}\right) = \sec(1) + \sec\left(\frac{1}{2}\right) + \sec\left(\frac{1}{3}\right) + \cdots$

$$\lim_{n\to\infty} \sec\left(\frac{1}{n}\right) = \sec\left(\lim_{n\to\infty} \frac{1}{n}\right) = \sec(0) = 1 \quad (\neq 0)$$

Thus, $\sum\limits_{n=1}^{\infty} \sec\left(\frac{1}{n}\right)$ <u>diverges</u> by the divergence test.

<u>Ex:</u> $\sum\limits_{n=1}^{\infty} \frac{1}{n(1+\ln n)}$

In this case we have $\lim\limits_{n\to\infty} \frac{1}{n(1+\ln n)} = 0$. So, what can we conclude from this?

<u>NOTHING!</u>

<u>Important Remark:</u>

The divergence test gives no information if $\lim\limits_{n\to\infty} a_n = 0$.

The series could converge or diverge!

Ex: Both $\sum_{n=1}^{\infty} \frac{1}{2^n}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ satisfy $\lim_{n\to\infty} a_n = 0$,

yet $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges while $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.