§6.3, 6.4 - Differentiating and Integrating Power Series

We can also obtain new power series by differentiating or integrating a power series term-by-term. These operations won't change the radius of convergence!

Theorem: If $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ with radius of convergence $R > 0$, then f is differentiable (hence continuous and integrable) for $x \in (a-R, a+R)$.

Moreover,

(i) $f'(x) = \sum_{n=1}^{\infty} c_n \cdot n (x-a)^{n-1}$

Note that differentiation kills the $n=0$ term (i.e., the constant term!)

(ii) $\int f(x)\,dx = \left(\sum_{n=0}^{\infty} \frac{c_n (x-a)^{n+1}}{n+1} \right) + C$

and both of these series have radius of convergence R — the same radius as f.

Note: While differentiating or integrating a power series won't change its radius, they may change the interval!

We need to re-check convergence at the endpoints!

Ex: Find a power series representation centred at $x=0$ and interval of convergence for each function $f(x)$.

(a) $f(x) = \dfrac{1}{(1-x)^2}$.

Solution: Note that

$$\frac{1}{(1-x)^2} = \left[\frac{1}{1-x}\right]' = \left[\sum_{n=0}^{\infty} x^n\right]' = \boxed{\sum_{n=1}^{\infty} n x^{n-1}}$$

Since $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n$ has radius of convergence $R=1$, so too does our new series. Thus, we at least have convergence for $x \in (-1,1)$. We check the endpoints.

$$x=1 \Rightarrow \sum_{n=1}^{\infty} n(1)^{n-1} = \sum_{n=1}^{\infty} n$$

Diverges by the divergence test!

$$x = -1 \Rightarrow \sum_{n=1}^{\infty} n(-1)^{n-1}$$ Diverges by the divergence test!

$\therefore$ The interval of convergence is $\boxed{I = (-1,1).}$

(b) $f(x) = \dfrac{1}{(1-x)^3}$

<u>Solution:</u> Notice that

$$\left(\frac{1}{1-x}\right)'' = \left(\frac{1}{(1-x)^2}\right)' = \frac{2}{(1-x)^3},$$

hence

$$\frac{2}{(1-x)^3} = \left(\frac{1}{1-x}\right)'' = \left(\sum_{n=0}^{\infty} x^n\right)'' = \left(\sum_{n=1}^{\infty} n x^{n-1}\right)' = \sum_{n=2}^{\infty} n(n-1)x^{n-2}$$

Radius $R = 1$

Thus, $\boxed{\dfrac{1}{(1-x)^3} = \displaystyle\sum_{n=2}^{\infty} \frac{n(n-1)}{2} x^{n-2}}$ with radius $R = 1$.

<u>Exercise:</u> Check convergence at $x = \pm 1$. You should find that $\boxed{I = (-1,1).}$

(c) $f(x) = \arctan(x)$

Solution: We'll start by finding a power series for $\frac{1}{1+x^2}$. Since $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$, we have

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

For convergence, we need

$|-x^2| < 1 \Rightarrow |x|^2 < 1$

$\Rightarrow |x| < 1 \quad (R=1)$

Thus,

$$\arctan(x) = \int \frac{1}{1+x^2}\, dx$$

$$= \int \sum_{n=0}^{\infty} (-1)^n x^{2n}\, dx$$

$$= \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \right) + C$$

We can find C by plugging in $x = 0$ (or in general, $x =$ centre):

$$\arctan(0) = \left(\sum_{n=0}^{\infty} \cancelto{=0}{\frac{(-1)^n 0^{2n+1}}{2n+1}} \right) + C \Rightarrow \underline{C = \arctan(0) = 0}.$$

Therefore,

$$\boxed{\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots}$$

The radius is still $R = 1$, so it converges for $x \in (-1,1)$.

Let's check the endpoints!

$$x = 1 \Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n 1^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

(converges by the AST!)

$$x = -1 \Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n (-1)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1}$$

(converges by the AST!)

-1 to odd power is just -1

Thus, the interval of convergence is $\boxed{I = [-1, 1]}$.

<u>Consequence:</u> Since $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$ for all $x \in [-1,1]$, let's see what happens when we substitute $x = 1$:

$$\arctan 1 = 1 - \frac{1^3}{3} + \frac{1^5}{5} - \frac{1^7}{7} + \cdots$$

$$\Rightarrow \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

$$\overset{\cdot 4}{\Rightarrow} \underline{\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \cdots}$$

(Whoa!!)

We can also use differentiation of power series to obtain a power series for e^x!

<u>Proposition</u>: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ for $x \in (-\infty, \infty)$.

<u>Proof</u>: In an earlier lesson, we used the ratio test to show that $y = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all $x \in (-\infty, \infty)$.

We will prove that $y = e^x$. Notice that

$$y' = \left(\sum_{n=0}^{\infty} \frac{x^n}{n!} \right)' = \sum_{n=1}^{\infty} \frac{n x^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} \overset{\text{Re-index!}}{=} \sum_{n=0}^{\infty} \frac{x^n}{n!} = y$$

The solutions to the DE $y' = y$ are $y = Ce^x$, $C \in \mathbb{R}$.

Since $y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ satisfies $y(0) = 1$, we have

$$y = Ce^x \Rightarrow 1 = y(0) = Ce^0 = C \Rightarrow y = e^x. \quad \blacksquare$$

Additional Exercises

1. Use integration to show that

$$\boxed{\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots}$$

with interval of convergence $I = (-1, 1]$.

2. Find a power series centred at $x=0$ for

$f(x) = \dfrac{x^2}{(3+x)^2}$ and its interval of convergence.

Solutions:

(a) First, note that

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$$

with radius of convergence $R = 1$. Hence,

$$\ln|1+x| = \int \frac{1}{1+x}\, dx$$

$$= \int \sum_{n=0}^{\infty} (-1)^n x^n \, dx$$

$$= \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} \right) + C$$

also with $R = 1$. We find C by setting $x = 0$:

$$\ln|1+0| = \sum_{n=0}^{\infty} \cancel{\frac{(-1)^n \, 0^{n+1}}{n+1}}^{=0} + C \Rightarrow C = \ln 1 = 0.$$

We therefore have

$$\ln|1+x| = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

with $R = 1$. We have convergence for $x \in (-1,1)$ and must separately check convergence at $x = \pm 1$:

$\underline{x = 1}$ $\Rightarrow$ $$\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$$ (converges by AST)

$\underline{x = -1}$ $\Rightarrow$ $$\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n (-1)^{n+1}}{n+1}$$

Always $= -1$

$$= -\underbrace{\sum_{n=0}^{\infty} \frac{1}{n+1}}_{\text{harmonic series}} \Rightarrow \text{divergent}$$

Thus, $\underline{I = (-1,1]}$, as required. (Finally, note that since $|1+x| = 1+x$ for $x \in (-1,1]$, we may simply write $\ln(1+x)$ instead of $\ln|1+x|$.)

(b) Note that $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ with radius $R = 1$.

$$\Rightarrow \frac{1}{(1-x)^2} = \left(\frac{1}{1-x}\right)' = \left(\sum_{n=0}^{\infty} x^n\right)' = \sum_{n=1}^{\infty} n x^{n-1} \quad \text{with } R = 1$$

Replace x with $\frac{-x}{3}$

$$\Rightarrow \frac{x^2}{(3+x)^2} = \frac{x^2}{3^2\left(1+\frac{x}{3}\right)^2} = \frac{x^2}{9} \cdot \frac{1}{\left(1-\left(\frac{-x}{3}\right)\right)^2}$$

$$= \frac{x^2}{9} \sum_{n=1}^{\infty} n\left(\frac{-x}{3}\right)^{n-1}$$

For convergence, need $\left|\frac{-x}{3}\right| < 1$
$\Rightarrow |x| < 3$
$\Rightarrow x \in (-3,3)$.

$$= \frac{x^2}{9} \sum_{n=1}^{\infty} \frac{n(-1)^{n-1} x^{n-1}}{3^{n-1}}$$

$$= \boxed{\sum_{n=1}^{\infty} \frac{n(-1)^{n-1} x^{n+1}}{3^{n+1}}}$$

Exercise: Check convergence at the endpoints of $(-3,3)$. You should find that the series diverges in each case, hence

$$\boxed{I = (-3,3).}$$