§5.5 — The Comparison Tests

We can compare series in much the same way that we compare improper integrals!

The Comparison Test

Suppose that $0 \le a_n \le b_n$ for all n sufficiently large.

(i) If $\sum b_n$ converges, then $\sum a_n$ converges.

(ii) If $\sum a_n$ diverges, then $\sum b_n$ diverges.

Note: If $\sum a_n$ converges or $\sum b_n$ diverges, the comparison test gives no information!

Ex: Do the following converge or diverge?

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^5+2}$

Solution: Note that $0 \le \dfrac{1}{n^5+2} \le \dfrac{1}{n^5}$ for all n.

Since $\sum_{n=1}^{\infty} \frac{1}{n^5}$ is a convergent p-series (here, $p=5>1$),

$\sum_{n=1}^{\infty} \frac{1}{n^5+2}$ converges by comparison.

(b) $\sum_{n=0}^{\infty} \frac{2^n}{3^n+n}$

Solution: We have $0 \leq \frac{2^n}{3^n+n} \leq \frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n$ and

$\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n$ is a convergent geometric series ($|r| = 2/3 < 1$).

Consequently, $\sum_{n=0}^{\infty} \frac{2^n}{3^n+n}$ converges by comparison.

(c) $\sum_{n=1}^{\infty} \frac{\sqrt{n^3+n+1}}{n^2}$

Solution: We have

$$\frac{\sqrt{n^3+n+1}}{n^2} \geq \frac{\sqrt{n^3}}{n^2} = \frac{n^{3/2}}{n^2} = \frac{1}{\sqrt{n}} \geq 0$$

and $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a divergent p-series (as $p = \frac{1}{2} \leq 1$).

Thus, $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^3+n+1}}{n^2}$ also <u>diverges</u> by comparison.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2(n)+1}{n^8}$

<u>Solution</u>: Since $0 \leq \cos^2(n) \leq 1$, we have

$$0 \leq \frac{\cos^2(n)+1}{n^8} \leq \frac{1+1}{n^8} = \frac{2}{n^8}.$$

The series $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^8} = 2\sum_{n=1}^{\infty} \frac{1}{n^8}$ converges, as it is 2 times a convergent p-series. As a result,

$\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2(n)+1}{n^8}$ <u>converges</u> by comparison.

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n-1}$

<u>Idea</u>: Try removing the "-1" in the denominator.

We have $\displaystyle\frac{1}{2^n-1} \geq \frac{1}{2^n} = \left(\frac{1}{2}\right)^n \geq 0$ for all n and

$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ is a convergent geometric series ...

... but this isn't helpful!

> Recall: Being larger than a convergent series or smaller than a divergent series tells us nothing!

We still guess that $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ converges, as $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges and $\frac{1}{2^n - 1} \approx \frac{1}{2^n}$ when n is large.

To make this precise, we'll need a new test!

The Limit Comparison Test (LCT)

Let $\sum a_n$ and $\sum b_n$ be series of positive terms, and let

$$\boxed{L = \lim_{n \to \infty} \frac{a_n}{b_n}}$$

If L exists and $0 < L < \infty$, then $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

Back to the example of $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$:

Let's try the LCT with $a_n = \frac{1}{2^n - 1}$ and $b_n = \frac{1}{2^n}$

We have

$$L = \lim_{n\to\infty} \frac{a_n}{b_n}$$

$$= \lim_{n\to\infty} \frac{\left(\frac{1}{2^n-1}\right)}{\left(\frac{1}{2^n}\right)}$$

$$= \lim_{n\to\infty} \frac{2^n}{2^n - 1}$$

$$= \lim_{n\to\infty} \frac{\cancel{2^n}}{\cancel{2^n}\left(1 - \underbrace{\tfrac{1}{2^n}}_{\to 0}\right)} = \frac{1}{1-0} = 1.$$

Since L exists and $0 < L < \infty$, the LCT implies that $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ and $\sum_{n=1}^{\infty} \frac{1}{2^n}$ either both converge or both diverge. Thus, since $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a convergent geometric series ($|r| = \frac{1}{2}$), $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ must <u>converge</u> too!

Ex: Does $\sum_{n=1}^{\infty} \frac{n}{n^2+6}$ converge or diverge?

[Previously we used the integral test to show this is divergent... but that took a lot of work!]

Solution: Let's try the LCT with $a_n = \frac{n}{n^2+6}$ and

$b_n = \frac{n}{n^2} = \frac{1}{n}$.

Tip: Use the most dominant term in the numerator and denominator to define b_n!

We have

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\frac{n}{n^2+6}\right)}{\left(\frac{1}{n}\right)}$$

$$= \lim_{n\to\infty} \frac{n^2}{n^2+6}$$

$$\stackrel{LH}{=} \lim_{n\to\infty} \frac{2n}{2n} = 1 \in (0,\infty).$$

By the LCT, since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (it's the harmonic series!), $\sum_{n=1}^{\infty} \frac{n}{n^2+6}$ must also diverge.

Ex: Does $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^6+4}}{2n^5+1}$ converge or diverge?

Solution: Use the LCT with $a_n = \dfrac{\sqrt{n^6+4}}{2n^5+1}$ and

$b_n = \dfrac{\sqrt{n^6}}{n^5} = \dfrac{n^3}{n^5} = \dfrac{1}{n^2}$. We have

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\frac{\sqrt{n^6+4}}{2n^5+1}\right)}{\left(\frac{1}{n^2}\right)}$$

$$= \lim_{n\to\infty} \frac{n^2\sqrt{n^6+4}}{2n^5+1}$$

$$= \lim_{n\to\infty} \frac{n^2\sqrt{n^6}\sqrt{1+4/n^6}}{2n^5+1}$$

$$= \lim_{n\to\infty} \frac{\cancel{n^5}\sqrt{1+4/n^6}}{\cancel{n^5}(2+1/n^5)} = \frac{\sqrt{1+0}}{2+0} = \frac{1}{2} \in (0,\infty).$$

Thus, by the LCT, since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series $(p=2>1)$, $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^6+4}}{2n^5+1}$ must converge too!

Ex: Suppose that $a_n > 0$ for all n and $\sum a_n$ converges. Show that $\sum \sin(a_n)$ must also converge.

Solution: Since $\sum a_n$ converges, $\lim_{n\to\infty} a_n = 0$ by the divergence test. Thus, for n sufficiently large, we'll have $0 < a_n < \pi/2$, meaning $\sin(a_n) > 0$. Now let's compare!

$$L = \lim_{n\to\infty} \frac{\sin(\cancelto{0}{a_n})}{\cancelto{0}{a_n}}$$

$$= \lim_{x\to 0^+} \frac{\sin(x)}{x} = 1 \in (0,\infty)$$

By the LCT, since $\sum a_n$ converges, $\sum \sin(a_n)$ also converges. ■

Extensions of the LCT

Assume that $\sum a_n$ and $\sum b_n$ are series with positive terms and let

$$L = \lim_{n\to\infty} \frac{a_n}{b_n}.$$

(i) If $L = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.

(ii) If $L = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Ex: Does $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ converge or diverge?

Solution: If we try the LCT with $a_n = \frac{1}{\ln n}$ and

and $b_n = \frac{1}{n}$, we get

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{\ln n}}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{n}{\ln n}$$

$$= \lim_{n\to\infty} \frac{1}{1/n}$$

$$= \lim_{n\to\infty} n = \infty$$

Since $L = \infty$ and $\sum b_n = \sum \frac{1}{n}$ diverges, by the

LCT, $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ diverges too!

[Alternatively, one could use a direct comparison!

Since $\ln n \le n$, we have $\frac{1}{\ln n} \geq \frac{1}{n} > 0$; and since $\sum \frac{1}{n}$ diverges, $\sum \frac{1}{\ln n}$ must diverge by comparison.]