§6.9 – The Binomial Series

You may be familiar with the binomial formula for expanding $(1+x)^m$ when m is a positive integer:

Binomial Formula: If m is a positive integer, then

$$(1+x)^m = \sum_{n=0}^{m} \binom{m}{n} x^n \quad \text{where} \quad \binom{m}{n} = \frac{m!}{n!(m-n)!}$$

Example:

$$(1+x)^3 = \binom{3}{0}x^0 + \binom{3}{1}x + \binom{3}{2}x^2 + \binom{3}{3}x^3$$

$$= \frac{3!}{0!(3-0)!} + \frac{3!}{1!(3-1)!}x + \frac{3!}{2!(3-2)!}x^2 + \frac{3!}{3!(3-3)!}x^3$$

$$= \frac{3!}{0!\cdot 3!} + \frac{3!}{1!\,2!}x + \frac{3!}{2!\,1!}x^2 + \frac{3!}{3!\,0!}x^3$$

$$= \boxed{1 + 3x + 3x^2 + x^3}$$

Using Maclaurin series, we can expand $(1+x)^m$ for any real number m, not just positive integers!

$$f(x) = (1+x)^m \qquad\qquad f(0) = 1$$

$$f'(x) = m(1+x)^{m-1} \qquad\qquad f'(0) = m$$

$$\Longrightarrow$$

$$f''(x) = m(m-1)(x+1)^{m-2} \qquad\qquad f''(0) = m(m-1)$$

$$f'''(x) = m(m-1)(m-2)(x+1)^{m-3} \qquad\qquad f'''(0) = m(m-1)(m-2)$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

In general, we have $f(0) = 1$ and for $n \in \mathbb{N}$,

$$f^{(n)}(0) = m(m-1)(m-2)\cdots(m-n+1)$$

Hence, we obtain the Maclaurin series

$$\boxed{1 + \sum_{n=1}^{\infty} \frac{m(m-1)(m-2)\cdots(m-n+1)}{n!} x^n}$$

Let's use the ratio test to find its radius of convergence!

$$L = \lim_{n\to\infty} \left| \frac{\dfrac{\cancel{m(m-1)(m-2)\cdots(m-n+1)}(m-n)}{(n+1)!} x^{n+1}}{\dfrac{\cancel{m(m-1)(m-2)\cdots(m-n+1)}}{n!} x^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{m-n}{n+1} \right| |x|$$

$$= |x|$$

We have $L<1 \iff |x|<1 \iff x \in (-1,1)$, so the Maclaurin series has radius 1 and converges at least for $x \in (-1,1)$.

Remarks:

1. Depending on m, the series may also converge at the endpoints, but this is complicated. You're not expected to know this for $(1+x)^m$.

2. When $m \in \mathbb{N}$, the Maclaurin series is just a polynomial,

since $m(m-1)(m-2)\cdots(m-n+1)=0$ when $m\in\mathbb{N}$ and $n>m$; hence it converges for all x. In fact, it just ends up being our usual binomial formula!

<u>Notation</u>: We adopt the "m choose n" notation for any $m\in\mathbb{R}$. For $m\in\mathbb{R}$, we define $\binom{m}{0}=1$ and

$$\binom{m}{n}=\frac{m(m-1)(m-2)\cdots(m-n+1)}{n!} \quad \text{for } n\geq 1.$$

When $m\in\mathbb{N}$, this is exactly $\frac{m!}{n!(m-n)!}$ — but "$m!$" isn't defined for $m\notin\mathbb{N}$. Writing $\frac{m(m-1)(m-2)\cdots(m-n+1)}{n!}$ instead allows us to make sense of $\binom{m}{n}$ for any $m\in\mathbb{R}$!

It turns out that $(1+x)^m$ is equal to its Maclaurin series for all x in the interval of convergence. We won't prove this here. Instead, let's summarize our results and explore some examples!

The Binomial Series: For any $m \in \mathbb{R}$ and all $x \in (-1,1)$,

$$(1+x)^m = \sum_{n=0}^{\infty} \binom{m}{n} x^n$$

where $\binom{m}{0} = 1$ and $\binom{m}{n} = \dfrac{m(m-1)(m-2)\cdots(m-n+1)}{n!}$ $(n \geq 1)$.

Ex: Find the Maclaurin series for each function.

(a) $f(x) = \dfrac{1}{\sqrt{1+x}}$

Solution:

$$\frac{1}{\sqrt{1+x}} = (1+x)^{-1/2} = \sum_{n=0}^{\infty} \binom{-1/2}{n} x^n$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})\cdots\overbrace{(-\frac{1}{2}-n+1)}^{= \frac{-(2n-1)}{2}}}{n!} x^n$$

$$= \boxed{1 + \sum_{n=1}^{\infty} \frac{(-1)^n 1\cdot 3\cdot 5\cdots(2n-1)}{2^n n!} x^n}$$

and this is valid for $\underline{|x| < 1}$.

(b) $g(x) = \dfrac{1}{\sqrt{1-x^2}}$

Solution:

Replace x in (a) with $-x^2$

$$\frac{1}{\sqrt{1-x^2}} = \frac{1}{\sqrt{1+(-x^2)}} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n\, 1\cdot 3\cdot 5\cdots(2n-1)}{2^n\cdot n!}(-x^2)^n$$

$$= 1 + \sum_{n=1}^{\infty} \frac{\overbrace{(-1)^n(-1)^n}^{=1}\, 1\cdot 3\cdot 5\cdots(2n-1)}{2^n\cdot n!}\, x^{2n}$$

$$= \boxed{1 + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n\cdot n!}\, x^{2n}}$$

and this is valid when $|-x^2| < 1 \iff |x|^2 < 1 \iff \underline{|x| < 1}$

(c) $h(x) = \arcsin x$

Solution: Integrate the series from (b)!

$$\arcsin x = \int \frac{1}{\sqrt{1-x^2}}\, dx$$

$$= \int \left(1 + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{2^n \cdot n!} x^{2n} \right) dx$$

$$= x + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{2^n \cdot n! \, (2n+1)} x^{2n+1} + C$$

And $C = 0$ since $\arcsin(0) = 0$.

$$= \boxed{x + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{2^n \cdot n! \, (2n+1)} x^{2n+1}}$$

Since integration doesn't affect the radius of convergence, this is valid for $|x| < 1$ (and, as mentioned earlier, we won't deal with endpoints for the binomial series!)