Big idea: Integrals arise when adding a continuum of tiny quantities. You can think of $\int_a^b f(x)\,dx$ like a big sum!

§1.4 - Average Values

Recall: The average value of real numbers $y_1, y_2, \ldots, y_n$ is

$$\boxed{y_{avg.} = \frac{y_1 + y_2 + \cdots + y_n}{n}}$$

add all quantities

divide by the number of quantities (the sample size)

Similarly, we can define the notion of an "average" for infinitely many quantities. Specifically, the average value of a continuous function $y = f(x)$ for $x \in [a,b]$ is

$$\boxed{f_{avg.} = \frac{1}{b-a}\int_a^b f(x)\,dx}$$

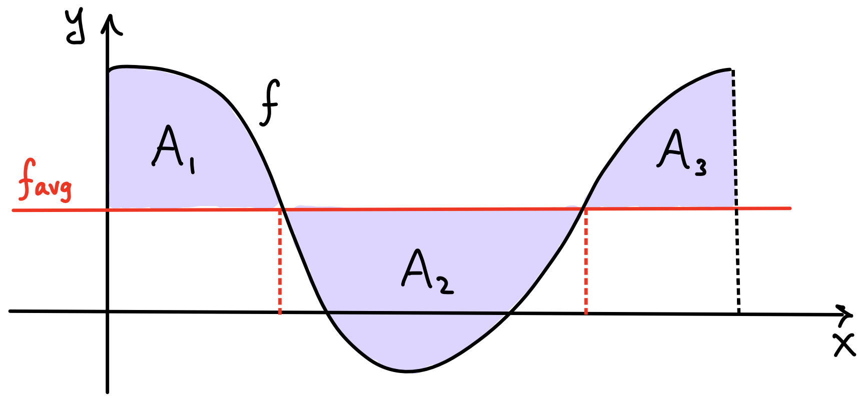
"Add" all quantities

Divide by the length of $[a,b]$

(similar to dividing by the sample size!)

Geometrically:

$$\boxed{\text{Area below } f_{avg} = \text{Area above } f_{avg}}$$



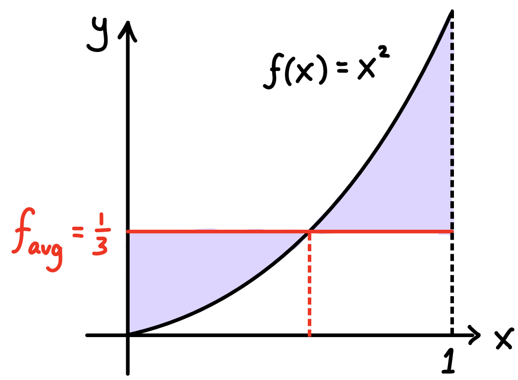
$$A_1 + A_3 = A_2$$

Ex: The average value of $f(x) = x^2$ for $x \in [0, 1]$ is

$$f_{avg.} = \frac{1}{1-0} \int_0^1 f(x)\, dx$$

$$= 1 \cdot \int_0^1 x^2\, dx$$

$$= \frac{1}{3}$$ Exercise from last time!



Notice: In both pictures above, there is (at least) one point $x = c$ where f attains its average value. When f is continuous, this will always occur!

The Average Value Theorem (AVT)

If $f(x)$ is continuous on $[a,b]$, then there exists a $c \in [a,b]$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\,dx$$

Proof: By the Extreme Value Theorem, f attains maximum and minimum values on $[a,b]$. That is, there exist m, M such that

$$m \leq f(x) \leq M \quad \text{for all } x \in [a,b]$$

where $m = f(c_1)$ and $M = f(c_2)$ for some $c_1, c_2 \in [a,b]$.

Thus, by our integral properties, we have

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

Dividing by $b-a$,

$$m \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq M,$$

hence

$$f(c_1) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq f(c_2).$$

Since f is continuous, by the Intermediate Value Theorem, there exists c between c_1 and c_2 such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx,$$

as required. $\blacksquare$

<u>Example</u>: Suppose f is continuous and $\displaystyle\int_2^5 f(x)\,dx = 12$.

Show that there exists $c \in [2,5]$ with $f(c) = 4$.

<u>Solution</u>: By the AVT, there exists $c \in [2,5]$ with

$$f(c) = f_{avg} = \frac{1}{5-2}\int_2^5 f(x)\,dx = \frac{1}{3}\cdot 12 = 4,$$

as desired.