§6.10 – Applications of Taylor Series

We'll explore 3 applications of Taylor series!

① We can find exact sums of certain infinite series using Taylor series!

Ex: We know

$$\ln|1+x| = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad \text{for } x \in (-1, 1].$$

Plugging in $x = 1$, we get

$$\boxed{\ln(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots}$$

We've just found the sum of the alternating harmonic series!

Ex: Find the exact sum of each series

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{\pi^{2n}}{n!} = 1 + \pi^2 + \frac{\pi^4}{2!} + \frac{\pi^6}{3!} + \cdots$

<u>Solution:</u> This series resembles $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

Indeed,

$$\sum_{n=0}^{\infty} \frac{\pi^{2n}}{n!} = \sum_{n=0}^{\infty} \frac{(\pi^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Bigg|_{x=\pi^2} = \boxed{e^{\pi^2}}$$

(b) $\sum_{n=0}^{\infty} \frac{n}{5^n}$

<u>Solution:</u> This looks like $\underbrace{\sum_{n=1}^{\infty} n x^n}_{\text{What function is this equal to??}}$ where $x = \frac{1}{5}$.

Note that $$\sum_{n=1}^{\infty} n x^n = x \left[\sum_{n=1}^{\infty} n x^{n-1} \right]$$

$$= x \left[\sum_{n=0}^{\infty} x^n \right]'$$

$$= x \left[\frac{1}{1-x} \right]' = \frac{x}{(1-x)^2},$$

and since $\sum_{n=0}^{\infty} x^n$ has radius of convergence $R=1$, so too does $\sum_{n=1}^{\infty} n x^n$ (hence will converge when $x = \frac{1}{5}$).

Thus,

$$\sum_{n=1}^{\infty} \frac{n}{5^n} = \sum_{n=1}^{\infty} n x^n \Big|_{x=\frac{1}{5}} = \frac{x}{(1-x)^2}\Big|_{x=\frac{1}{5}} = \boxed{\frac{5}{16}}$$

(c) $\sum_{n=2}^{\infty} \frac{(-1)^n n(n-1)}{4^n}$

<u>Solution:</u> This looks like $\sum_{n=2}^{\infty} n(n-1)x^n$ where $x = \frac{-1}{4}$.

Note that

$$\sum_{n=2}^{\infty} n(n-1)x^n = x^2 \sum_{n=2}^{\infty} n(n-1)x^{n-2}$$

$$= x^2 \left[\sum_{n=0}^{\infty} x^n\right]''$$

$$= x^2 \left[\frac{1}{1-x}\right]'' = \frac{2x^2}{(1-x)^3}.$$

Thus, $\sum_{n=2}^{\infty} \frac{(-1)^n n(n-1)}{4^n} = \sum_{n=2}^{\infty} n(n-1)x^n \Big|_{x=\frac{-1}{4}}$

$$= \left. \frac{2x^2}{(1-x)^3} \right|_{x=-\frac{1}{4}}$$

$$= \frac{2\left(-\frac{1}{4}\right)^2}{\left(1+\frac{1}{4}\right)^3} = \boxed{\frac{8}{125}}$$

② We can use Taylor series to calculate indeterminate limits without L'Hopital's Rule!

Ex: Use Taylor series to evaluate the following limits (and NOT L'Hopital's rule).

(a) $\lim\limits_{x \to 0} \frac{e^x - 1}{x}$

Solution:

$$\lim_{x \to 0} \frac{e^x - 1}{x} = \lim_{x \to 0} \frac{\overbrace{\left[\cancel{1} + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right]}^{\text{Replace } e^x \text{ with Maclaurin series!}} - \cancel{1}}{x}$$

$$= \lim_{x \to 0} \frac{x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots}{x}$$

$$= \lim_{x \to 0} \left(1 + \frac{x}{2!} + \frac{x^2}{3!} + \cdots\right)$$

$$= 1 + 0 + 0 + \dots$$

$$= \boxed{1}$$

(b) $\lim_{x \to 0} \frac{x^3}{\sin(x) - x}$

Solution:

$$\lim_{x \to 0} \frac{x^3}{\sin(x) - x} = \lim_{x \to 0} \frac{x^3}{\left(\cancel{x} - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right) - \cancel{x}}$$

$$= \lim_{x \to 0} \frac{x^3}{-\frac{x^3}{3!} + \frac{x^5}{5!} - \dots} \quad \begin{matrix} \div x^3 \\ \div x^3 \end{matrix}$$

$$= \lim_{x \to 0} \frac{1}{-\frac{1}{3!} + \frac{x^2}{5!} - \dots}$$

$$= \frac{1}{-\frac{1}{6} + 0 - 0 + \dots}$$

$$= \boxed{-6}$$

(c) $\lim_{x \to 0} \frac{1 - \cos(2x)}{e^x - x - 1}$

Solution:

$$\lim_{x\to 0} \frac{1-\cos(2x)}{e^x - x - 1} = \lim_{x\to 0} \frac{\cancel{1} - \left(\cancel{1} - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots\right)}{\left(\cancel{1} + \cancel{x} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) - \cancel{x} - \cancel{1}}$$

$$= \lim_{x\to 0} \frac{\frac{2^2x^2}{2!} - \frac{2^4x^4}{4!} + \frac{2^6x^6}{6!} - \cdots}{\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots} \qquad \begin{matrix} \div x^2 \\ \div x^2 \end{matrix}$$

$$= \lim_{x\to 0} \frac{\frac{2^2}{2!} - \frac{2^4x^2}{4!} + \frac{2^6x^4}{6!} - \cdots}{\frac{1}{2!} + \frac{x}{3!} + \frac{x^2}{4!} + \cdots}$$

$$= \frac{\frac{2^2}{2!} - 0 + 0 - \cdots}{\frac{1}{2!} + 0 + 0 + \cdots}$$

$$= \frac{\frac{4}{2}}{\frac{1}{2}} = \boxed{4}$$

③ We can use Taylor series to evaluate impossible integrals (as series)!

Ex: (a) Use Maclaurin series to evaluate $\int e^{-x^2}\,dx$ as a series.

Note: $\int e^{-x^2}\,dx$ is impossible to evaluate using our earlier integration methods, as e^{-x^2} doesn't have an antiderivative in terms of our elementary functions.

Solution: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, hence we have

$$\Rightarrow e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$

$$\Rightarrow \int e^{-x^2}\,dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}\,dx = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!\,(2n+1)} + C}$$

(b) Express $\int_0^1 e^{-x^2}\,dx$ as a series and approximate its value with error at most $\frac{1}{100}$.

Solution:

$$\int_0^1 e^{-x^2}\,dx = \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!\,(2n+1)}\right]_0^1$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(2n+1)}$$

converges by AST since $b_n = \frac{1}{n!(2n+1)}$ decreases to 0.

$$= \underbrace{\frac{1}{0!\cdot 1} - \frac{1}{1!\cdot 3} + \frac{1}{2!\cdot 5} - \frac{1}{3!\cdot 7}}_{S_3} - \frac{1}{4!\cdot 9} + \cdots$$

$< \frac{1}{100}$

Thus, by the Alternating Series Estimation Theorem,

$$\int_0^1 e^{-x^2}\,dx \approx S_3 = \boxed{\frac{26}{35} \approx 0.74286}$$

with $|\text{error}| \leq b_{n+1} = \frac{1}{4!\cdot 9} < \frac{1}{100}$.

End of MATH 138