§5.7 – Alternating Series

A series $\sum a_n$ is said to be alternating if the terms a_n alternate between positive and negative values.

Ex: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$ is alternating (it is known as the alternating harmonic series.)

Ex: $1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} - \frac{1}{7} + \cdots$ is not considered to be alternating (sign must change from each term to the next!)

[Look for things like $(-1)^{n+1}$, $(-1)^n$, $\cos(n\pi)$, etc.]

Below is a very simple test that can be used to show that certain alternating series converge.

The Alternating Series Test (AST)

Consider the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n = b_1 - b_2 + b_3 - b_4 + \cdots$$

where $b_n > 0$ for all n. If

(i) $\{b_n\}$ is a decreasing sequence, and

(ii) $\lim_{n\to\infty} b_n = 0$,

then $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ converges.

Let's apply the AST to the alternating harmonic series, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

Here we have $b_n = \frac{1}{n}$. Note that

(i) $\left\{\frac{1}{n}\right\}$ is a decreasing sequence, and

(ii) $\lim_{n\to\infty} \frac{1}{n} = 0$, hence, by the AST, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges!

Remarks about the AST:

1. AST also applies to series of the form $\sum_{n=1}^{\infty} (-1)^n b_n$, $\sum_{n=0}^{\infty} (-1)^{n-1} b_n$, etc. Just make sure the series is alternating!

2. If only (i) fails (i.e., $\{b_n\}$ is non-decreasing), the AST provides no information.

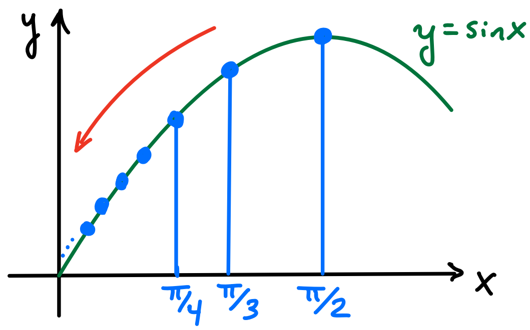
3. However, if (ii) fails (i.e., $\lim_{n\to\infty} b_n \neq 0$), then $\lim_{n\to\infty} (-1)^{n+1} b_n$ DNE, hence $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ diverges by the divergence test.

More Examples!

(a) $\sum_{n=2}^{\infty} (-1)^n \sin\left(\frac{\pi}{n}\right) = \sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{3}\right) + \sin\left(\frac{\pi}{4}\right) - \cdots$

Try the AST with $b_n = \sin(\pi/n)$. Note that

(i) $\{b_n\}$ is decreasing, as seen in the graph on the right.



[Alternatively, $f(x) = \sin(\pi/x)$ is decreasing since $f'(x) = \frac{-\pi}{x^2}\cos(\pi/x) < 0$.]

(ii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \sin\left(\frac{\pi}{n}\right) = \sin 0 = 0$.

Hence $\sum_{n=2}^{\infty} (-1)^n \sin\left(\frac{\pi}{n}\right)$ converges by the AST.

(b) $\sum_{n=1}^{\infty} (-1)^n e^{1/n} = -e^1 + e^{1/2} - e^{1/3} + e^{1/4} - e^{1/5} + \cdots$

Try AST with $b_n = e^{1/n}$. Note that

(i) $\{b_n\}$ is decreasing, since $f(x) = e^{1/x}$ has derivative

$f'(x) = \frac{-1}{x^2} e^{\frac{1}{x}} < 0$ everywhere.

(ii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} e^{1/n} = e^0 = 1$

uh oh... can't use AST! However, we can use the divergence test!

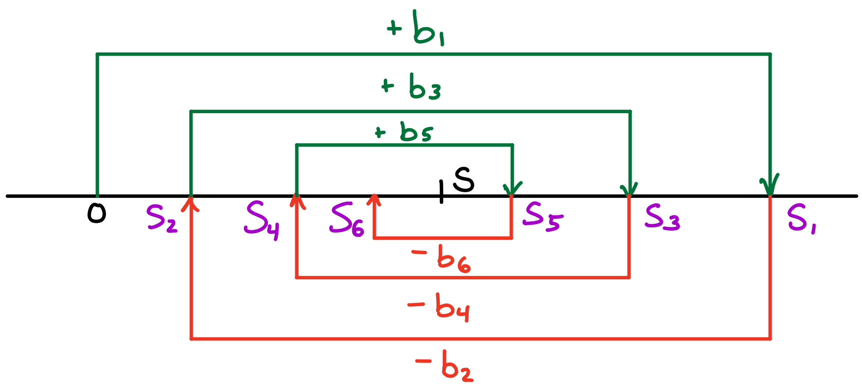
Since $\lim_{n\to\infty} (-1)^n e^{1/n}$ DNE (it jumps between ≈ 1 and ≈ -1),

$\sum_{n=1}^{\infty} (-1)^n e^{1/n}$ diverges by the divergence test.

Why does the AST work?

Suppose that $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ converges by the AST so

$b_n > 0$, $\{b_n\}$ is decreasing, and $\lim_{n\to\infty} b_n = 0$



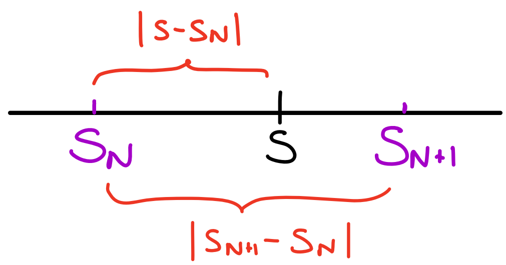
Since the terms b_n are decreasing, the partial sums

$$S_1, S_2, S_3, S_4, \ldots$$

"spiral inward" on the number line. The distance between the S_n's approaches 0 (since $\lim_{n\to\infty} b_n = 0$), meaning that the S_n's approach some limit S. This S is the sum of the series!

Application: Estimating Sums

From our "picture proof" of the the AST, the sum S lies between any S_N and S_{N+1}.



Thus, if we approximate S using a partial sum, S_N, the error $R_N = S - S_N$ satisfies

$$|R_N| = |S - S_N| \le |S_{N+1} - S_N|$$

$$= \underbrace{\left| \sum_{n=1}^{N+1} (-1)^{n+1} b_n - \sum_{n=}^{N} (-1)^{n+1} b_n \right|}_{\text{Everything except } (N+1)^{\text{th}} \text{ term cancels!}} = |\pm b_{N+1}| = b_{N+1},$$

Thus, we have the following result:

Alternating Series Estimation Theorem

If $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ converges by the AST, then the error in approximating the sum S by S_N satisfies

$$|R_N| = |S - S_N| \leq b_{N+1}$$

Ex: The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges by the AST.

(a) Estimate the size of the error when we use

$$S_4 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = 0.58\overline{3}$$

to approximate the sum, S.

Solution: Since $b_n = \frac{1}{n}$, the error satisfies

$$|R_4| \leq b_{4+1} = b_5 = \frac{1}{5} \quad (\text{or } 0.2)$$

(Note: Since $|R_4| = |S - S_4| \leq 0.2$, we have

$$-0.2 \le S - S_4 \le 0.2$$

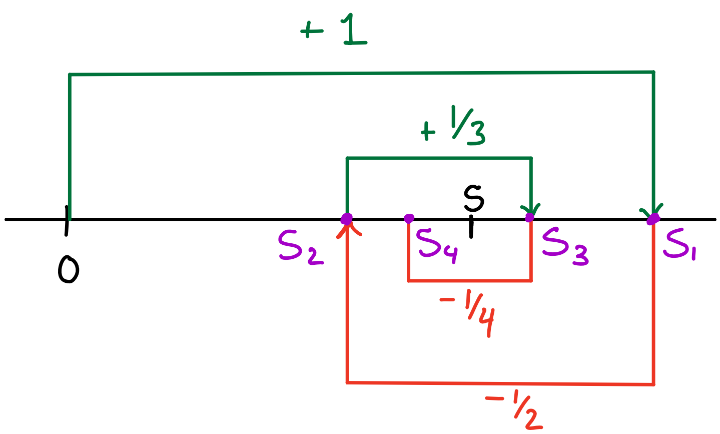
$$\Rightarrow \underset{+0.58\bar{3}}{-0.2} \le \underset{+0.58\bar{3}}{S - 0.58\bar{3}} \le \underset{+0.58\bar{3}}{0.2}$$

$$\Rightarrow \underline{0.38\bar{3} \le S \le 0.78\bar{3}}$$

)

(b) Is S_4 an overestimate or underestimate of the sum, S?

Solution:



Since the final term in $S_4 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$ is negative, the partial sum S_4 will underestimate S.

(So we can actually say $0.58\bar{3} \le S \le 0.78\bar{3}$)

Ex: The series $S = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$ converges by the AST.

(a) How many terms N will guarantee that the partial sum S_N approximates S with $|\text{Error}| \le \frac{1}{1000}$?

Solution: With $b_n = \frac{1}{n!}$, the error satisfies

$$|R_N| \le b_{N+1} = \frac{1}{(N+1)!}$$

Thus, we want

$$\frac{1}{(N+1)!} \le \frac{1}{1000} \iff (N+1)! \ge 1000$$

It will be easiest to check small values of N:

N	3	4	5	6
$(N+1)!$	24	120	720	5040

≥ 1000

Thus we will need at least $N=6$ terms!

(b) Is S_{100} an overestimate or underestimate of the overall sum, S?

Solution: Note that the final term in S_{100} is

$$\frac{(-1)^{100}}{100!} = \frac{1}{100!},$$

which is positive. This means S_{100} will overestimate S.