§5.8 – Absolute vs. Conditional Convergence

Some series NEED a mix of positive and negative terms in order to converge...

e.g. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges by the AST, but

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$$ (the harmonic series) diverges.

... while others may converge even if *all* terms have the same sign.

e.g. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ converges by the AST, and *also*

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$$ converges by the p-series test.

It will be important to distinguish between these types of convergence.

Definition: A series $\sum a_n$ is said to

(i) converge absolutely if $\sum |a_n|$ converges.

(ii) converge conditionally if $\sum a_n$ converges but $\sum |a_n|$ diverges.

Using our new terminology, we can say that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ converges absolutely, while $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges conditionally.

The following result shows that absolute convergence is even stronger than regular convergence.

Theorem: If $\sum |a_n|$ converges, then $\sum a_n$ converges as well. That is, any absolutely convergent series is convergent

Proof: Suppose that $\sum |a_n|$ converges, hence so too does $\sum 2|a_n|$. We note that

Add $|a_n|$ to all sides

$$\left\{ \begin{array}{l} -|a_n| \le a_n \le |a_n| \\ 0 \le a_n + |a_n| \le \underbrace{2|a_n|}_{\text{terms of a convergent series.}} \end{array} \right.$$

So $\sum (a_n + |a_n|)$ converges by the comparison test.

Thus, since $\sum a_n = \underbrace{\sum (a_n + |a_n|)}_{\text{convergent}} - \underbrace{\sum |a_n|}_{\text{convergent}}$,

$\sum a_n$ must converge too! ■

This is useful because sometimes it's easier to show that a series converges absolutely!

Ex: Does $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n^3)}{n^3}$ converge or diverge?

Can't use integral test, comparison test, or LCT because $\frac{\sin(n^3)}{n^3}$ is sometimes negative!

Solution: Let's check absolute convergence!

For $\displaystyle\sum_{n=1}^{\infty} \left|\frac{\sin(n^3)}{n^3}\right|$, we know $0 \le \left|\frac{\sin(n^3)}{n^3}\right| \le \frac{1}{n^3}$

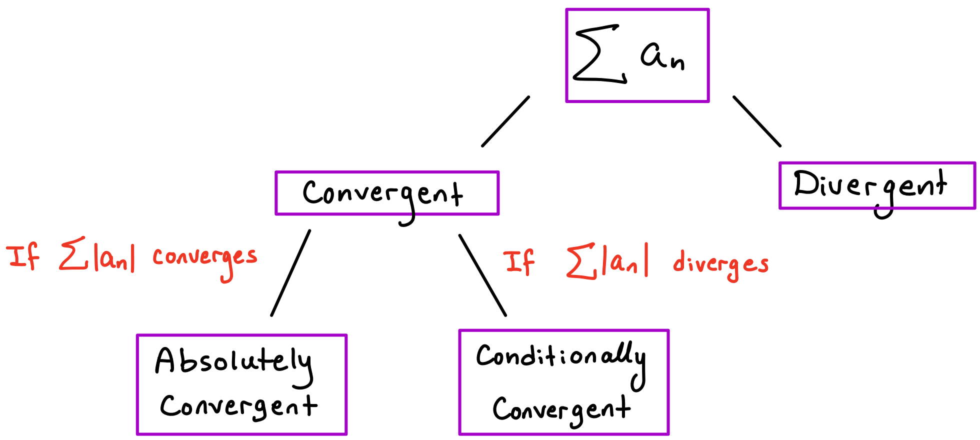
positive terms $\Rightarrow$ integral, comparison, LCT unlocked!

and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a convergent p-series $(p=3>1)$, so

$\displaystyle\sum_{n=1}^{\infty} \left|\frac{\sin(n^3)}{n^3}\right|$ converges by the comparison test. Thus,

$\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n^3)}{n^3}$ converges absolutely, hence <u>converges</u>.

The following chart summarizes all convergence possibilities:



<u>Ex:</u> Determine whether each series below converges absolutely, converges conditionally or diverges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n^2+1}}$

<u>Solution:</u> First note that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^2+1}}$ converges by the AST, as the terms $b_n = \dfrac{1}{\sqrt{n^2+1}}$ are decreasing and $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\sqrt{n^2+1}} = 0.$

To determine if the series converges absolutely, we look at

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{\sqrt{n^2+1}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$$

Using the LCT with $a_n = \dfrac{1}{\sqrt{n^2+1}}$ and $b_n = \dfrac{1}{\sqrt{n^2}} = \dfrac{1}{n}$, we have ...

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\frac{1}{\sqrt{n^2+1}}\right)}{\left(\frac{1}{n}\right)}$$

$$= \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}}$$

$$= \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{\sqrt{n^2}}\sqrt{1+\frac{1}{n^2}}}$$

$$= \frac{1}{\sqrt{1+0}} = 1 \in (0,\infty)$$

By the LCT, since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (it's the harmonic series), $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$ must also diverge.

Thus, $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^2+1}}$ does <u>not</u> converge absolutely.

<u>Conclusion:</u> $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^2+1}}$ <u>converges conditionally.</u>

(b) $\sum_{n=1}^{\infty} (-1)^n \frac{3^n}{1+2^n}$

<u>Solution:</u> The series is alternating, hence the AST may be tempting. Unfortunately, the terms $b_n = \frac{3^n}{1+2^n}$

don't tend to 0 as $n \longrightarrow \infty$. Indeed,

$$\lim_{n\to\infty} \frac{3^n}{1+2^n} = \lim_{n\to\infty} \frac{3^n}{2^n}\left(\frac{1}{\frac{1}{2^n}+1}\right)$$

$$= \lim_{n\to\infty} \underbrace{\left(\frac{3}{2}\right)^n}_{\longrightarrow\infty} \cdot \underbrace{\left(\frac{1}{\frac{1}{2^n}+1}\right)}_{\longrightarrow \frac{1}{0+1}=1} = \infty$$

So the AST doesn't apply ... but we can instead use

the divergence test: $\lim_{n\to\infty} \underbrace{(-1)^n}_{\text{oscillates}} \cdot \underbrace{\frac{3^n}{1+2^n}}_{\longrightarrow\infty}$ DNE, hence

$\sum_{n=1}^{\infty} (-1)^n \cdot \frac{3^n}{1+2^n}$ diverges.

Additional Exercises

Ex: True or False?

(a) If $\sum a_n$ converges, then $\sum (-1)^n a_n$ converges.

Solution: False. Let $a_n = \frac{(-1)^n}{n}$. Then $\sum a_n$ converges by the AST, but $\sum (-1)^n a_n = \sum \frac{(-1)^{2n}}{n} = \sum \frac{1}{n}$, the harmonic series, which diverges.

(b) If $\sum e^{a_n}$ converges, then $\sum a_n$ diverges.

Solution: True. We'll prove the contrapositive.

If $\sum a_n$ converges, then $\lim\limits_{n\to\infty} a_n = 0$ by the divergence test. Hence,

$$\lim_{n\to\infty} e^{a_n} = e^{\lim_{n\to\infty} a_n} = e^0 = 1 \qquad (\neq 0)$$

and so $\sum e^{a_n}$ diverges by the divergence test. ∎

(c) If $\sum a_n$ converges absolutely and $\{b_n\}$ is bounded, then $\sum a_n b_n$ converges absolutely

Solution: True. Since $\{b_n\}$ is bounded, there is a constant M such that $|b_n| \le M$ for all n. Consequently,

$$0 \le |a_n b_n| = |a_n| \cdot |b_n| \le M|a_n|$$

Since $\sum |a_n|$ converges, $\sum M|a_n|$ converges too, and so $\sum |a_n b_n|$ converges by comparison. That is, $\sum a_n b_n$ converges absolutely. $\blacksquare$