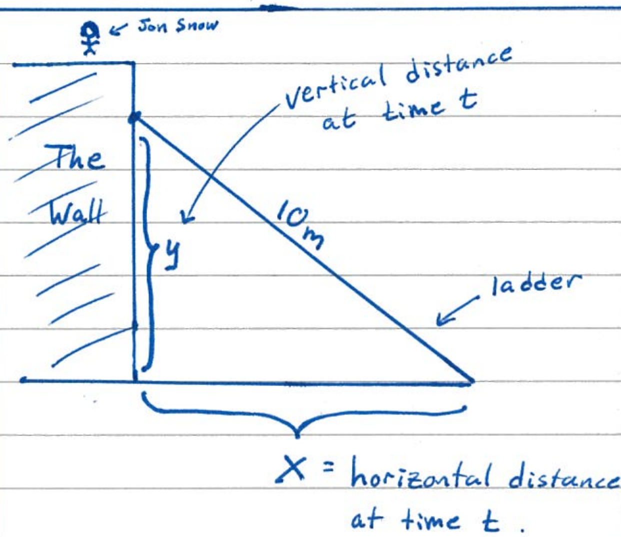
Related Rates (Not in textbook)

Idea: Suppose x and y change according to time t. If x and y are related, and we know $\frac{dx}{dt}$ at some time t, we can find $\frac{dy}{dt}$.

Ex: A ladder 10m long sits against a vertical wall. If the bottom of the ladder slips away at 1m/s, how fast is the top of the ladder sliding down the wall when the bottom is 6m away?

Steps for solving related rates problems	Solution to the above example.
(1) Draw a picture if possible. Use variables to represent any relevant quantities.	
(2) Find an equation that relates the known and unknown quantities. List any relevant information from the question.	The ladder makes a right triangle, so $x^2+y^2=10^2$. We also know that at this moment in time, $x(t)=6$, so $y(t)=\sqrt{100-6^2}=\sqrt{64}=8$. Furthermore, $\frac{dx}{dt}=1$.

(3) Differentiate the equation with respect to time t.

$x^2+y^2=100$, so

$2x\cdot\frac{dx}{dt}+2y\cdot\frac{dy}{dt}=0$ (implicit diff.)

Solve for the unknown quantity.

Know: $x=6$, $y=8$, $dx/dt=1$.
Want: dy/dt.

$2\cdot6\cdot1+2\cdot8\cdot\frac{dy}{dt}=0$

$\Rightarrow \frac{dy}{dt}=\frac{-12}{16}=-\frac{3}{4}$ m/s.

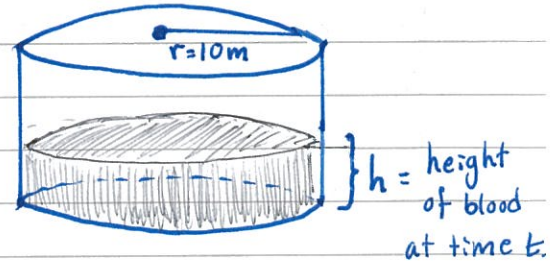
Write a concluding sentence.

The ladder is falling at a rate of 3/4 m/s.

Ex: A cylinder of radius 10m is filling with blood. If the height of the blood is increasing at a rate of 0.5 m/s, how fast is the volume of blood increasing?

Solution: Follow the above steps:

(1) Here's a picture of the scenario.



(2) Let V = volume of blood at time t.

Then $V=\pi r^2\cdot h=100\pi h$. We know $\frac{dh}{dt}=0.5$.

(3) $\dfrac{dV}{dt} = 100\pi \cdot \dfrac{dh}{dt} = 100\pi(0.5) = 50\pi \ m^3/s.$

$\boxed{\text{The volume is increasing at a rate of } 50\pi \ m^3/s}$

Ex: A snowball is melting so that its radius is decreasing at a rate of 0.5 cm/h.

How fast is the volume of the snowball decreasing when the radius is 6cm?

Solution: We don't really need a picture, but here's one anyway.

Let V = volume of snowball at time t,

r = radius of snowball at time t.

Recall that $V = \dfrac{4}{3}\pi r^3$. (volume of a sphere)

We know that $r = 6cm$ and $\dfrac{dr}{dt} = -0.5 cm/h$.

Thus, $\dfrac{dV}{dt} = \dfrac{4}{\cancel{3}} \cdot \pi \cdot \cancel{3} r^2 \cdot \dfrac{dr}{dt}$

$= 4\pi r^2 \cdot \dfrac{dr}{dt}$

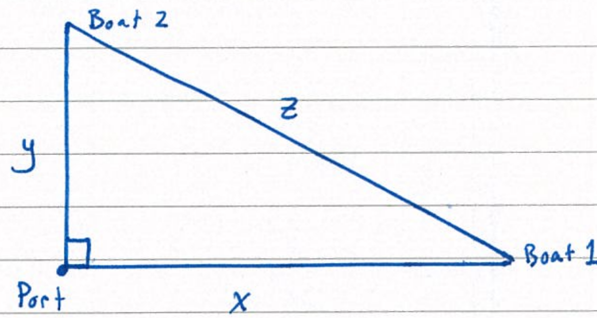
$= 4\pi(6)^2 \cdot (-0.5) = -72\pi \ cm^3/h.$

$\boxed{\text{The volume is decreasing at } 72\pi \ cm^3/h \text{ at this point in time}}$

Ex: Boat 1 leaves port at noon travelling east at 100 km/h. Boat 2 leaves the same port at 3pm travelling north at 150 km/h.

How fast is the distance between them changing at 5pm.

Solution: Here's the picture:



t = time in hours

x = Kilometers travelled by boat 1 at time t

y = Kilometers travelled by boat 2 at time t

z = distance (in km) between the boats at time t.

We want to know dz/dt at $t=5$.

At this time, $x = 100 \cdot 5 = 500$ km
$y = 150 \cdot 2 = 300$ km.
Boat 2 has only been out for 2 hours!

Since $x^2+y^2=z^2$, at $t=5$, $z = \sqrt{500^2+300^2} = \sqrt{34000} = 100\sqrt{34}$.

We also know $dx/dt = 100$, $dy/dt = 150$.

Differentiating $x^2+y^2=z^2$, we get $2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} = 2z\dfrac{dz}{dt}$.

$$\Rightarrow 2\cdot 500\cdot 100 + 2\cdot 300\cdot 150 = 2\cdot 100\sqrt{34}\cdot\frac{dz}{dt}$$

$$\Rightarrow \frac{dz}{dt} = \frac{950}{\sqrt{34}}.$$

The distance is changing at a rate of $\frac{950}{\sqrt{34}}$ km/h at 5pm.