# Optimization (Not in textbook)

Let's use the techniques of §7.2 to solve some real world problems!

**Ex:** A farmer has 800m of fencing to build a pen. The pen is rectangular and one side is against a river (and hence doesn't require fencing). What dimensions of the pen enclose the largest area?

| Steps for solving optimization problems | Solution to the above example. |
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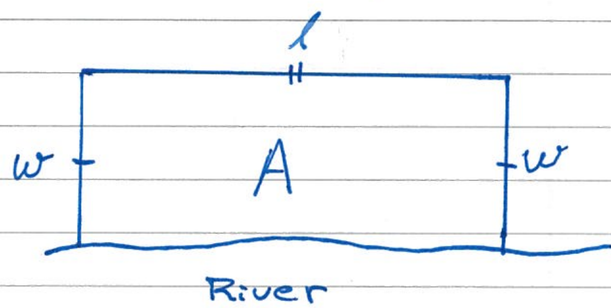
| (1) Draw a picture if possible<br><br>Use variables to represent any relevant quantities. | <br><br>$l$ = length of pen<br>$w$ = width of pen.<br>$A$ = area of pen |
| (2) Identify the quantity to be maximized / minimizd and find expression for it.<br><br>Write down any other conditions given in the question. | We're asked to maximize $A$.<br><br>Expression: $A = l \cdot w$.<br><br>Other Conditions: $l + 2W = 800$. |

(3) Use the equations in (2) to express the target function as a function with only 1 variable.

Since $l + 2w = 800$, we have that $l = 800 - 2w$.

Thus, $A = l \cdot w$
$= (800 - 2w) \cdot w$
$= 800w - 2w^2$

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(4) Determine the domain of that variable.

(check for any extreme situations)

The pen could have "no length" so $l = 0$. In this case, $2w = 800$, so $w = 400$.

The other extreme is a pen with "no width", so $w = 0$.

Thus, domain is $0 \le w \le 400$.

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(5) Maximize or minimize the target function on this domain.

(ie, find global extrema!)

We're maximizing $A = 800w - 2w^2$ on $[0, 400]$.

$A' = 800 - 4w \Rightarrow$ derivative exists everywhere

$A' = 0 \Rightarrow 800 = 4w$
$\Rightarrow w = \frac{800}{4} = 200$.

Our only critical point is $w = 200$.

$A(0) = 0$
$A(200) = 80000 \leftarrow$ global max.
$A(400) = 0$

Finally, provide a concluding sentence stating exactly what was asked.

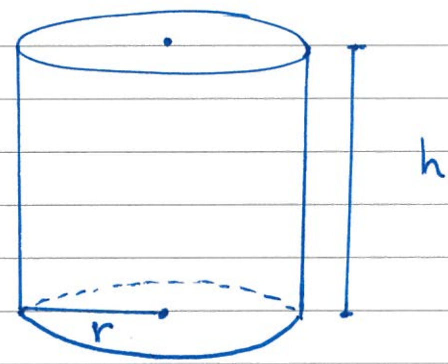
$\boxed{\text{Area is maximized when } w = 200\text{m and } l = 800 - 2w = 400\text{m.}}$

<u>Ex</u>: Suppose we have $300\,cm^2$ of tin and wish to make the largest cylindrical can possible. What should the dimensions of the can be? How much <u>liquid</u> could the can hold?

<u>Solution:</u> Follow the 5 steps as before.

(1) Here's a picture of the situation.

$h$ = height of can,
$r$ = radius of the base,
$V$ = volume of can.




(2) We're maximizing the volume

$$V = \pi \cdot r^2 \cdot h$$

Other constraints: $\underbrace{\pi r^2}_{\text{area of top}} + \underbrace{\pi r^2}_{\text{area of bottom}} + \underbrace{2\pi r \cdot h}_{\text{area of side}} = 300$

$$\Rightarrow 2\pi r^2 + 2\pi r h = 300$$

(3) From $2\pi r^2 + 2\pi r h = 300$, we get

$$2\pi r h = 300 - 2\pi r^2 \Rightarrow h = \frac{300 - 2\pi r^2}{2\pi r}$$

So,
$$\begin{aligned} V &= \pi r^2 h \\ &= \pi r^2 \left(\frac{300 - 2\pi r^2}{2\pi r}\right) \\ &= 150 r - \pi r^3 \end{aligned}$$

(now it's a function of 1 variable!)

(4) What is the domain of this function?

Clearly $r \geq 0$. The other extreme is when the can has "no height", so its area is $2\pi r^2$.

We then have $2\pi r^2 \leq 300 \Rightarrow r^2 \leq \frac{300}{2\pi} = \frac{150}{\pi}$

$$\Rightarrow r \leq \sqrt{\frac{150}{\pi}}.$$

The domain is $\left[0, \sqrt{\frac{150}{\pi}}\right]$.

(5) We're maximizing $V = 150r - \pi r^3$ on $\left[0, \sqrt{\frac{150}{\pi}}\right]$.

$V' = 150 - 3\pi r^2 \Rightarrow$ derivative exists everywhere.

$$V' = 0 \Rightarrow 150 = 3\pi r^2$$
$$\Rightarrow r^2 = \frac{150}{3\pi} = \frac{50}{\pi}$$
$$\Rightarrow r = \sqrt{\frac{50}{\pi}}$$

(This is our only critical point.)

Finally, $V(0) = 0$

$V\left(\sqrt{\frac{50}{\pi}}\right) \approx 398.9$ ← global max.

$V\left(\sqrt{\frac{150}{\pi}}\right) = 0$

$$\boxed{\begin{array}{l}\text{The volume of the largest can is } \approx 398.9\,\text{cm}^3. \\ \text{It occurs when } r = \sqrt{\frac{50}{\pi}}\ \text{cm and } h = \frac{300 - 2\pi r^2}{2\pi r} \approx 7.98\,\text{cm}.\end{array}}$$