Chapter 2 - Algebra Review (Hey, it rhymes!)

§2.1 - Polynomials

polynomial = finite sum of terms in which all variables have whole number exponents.

exponent

Ex: $3x^4 + 2x + 1$

coefficients variable constant

We can **add** polynomials by combining like terms:..

Ex: $(2x^2 - 1) + (x^3 - x^2 + 2) = x^3 - x^2 + 1.$

or **multiply** by distributing and adding exponents:

Ex: $(3 + x)(x^2 - x + 2) = (3x^2 - 3x + 6) + (x^3 - x^2 + 2x)$
$= x^3 + 2x^2 - x + 6.$

Ex: $(x+1)^3 = (x+1)(x+1)(x+1)$
$= (x^2 + x + x + 1)(x+1)$
$= (x^2 + 2x + 1)(x+1)$
$= x^3 + x^2 + 2x^2 + 2x + x + 1$
$= x^3 + 3x^2 + 3x + 1.$

§2.2 - Factoring

- Process of breaking-up polynomial into product of smaller polynomials.

Start by removing common factors

Ex: $8x^2 + 2x = 2x(4x+1)$

Factoring Quadratics: A quadratic is a polynomial of the form $Ax^2 + Bx + C$.

Sometimes we can guess at a factorization

Ex: $x^2 + 2x + 1 = (x+1)(x+1) = (x+1)^2$

sum $= B = 2$, product $= C = 1$.

Ex: $x^2 - 3x + 2 = (x-2)(x-1)$

sum $= B = -3$, product $= C = 2$.

When guessing is too tough, there is a formula that can be used.

Quadratic Formula: The solutions to $Ax^2 + Bx + C = 0$ are $x = \dfrac{-B \pm \sqrt{B^2 - 4AC}}{2A}$. These are called the roots.

Note: If $B^2-4AC < 0$, then the quadratic is called irreducible and cannot be factored.

Ex: $x^2-10x+21$.

By the quadratic formula, the roots are

$$x = \frac{-(-10) \pm \sqrt{(-10)^2-4(1)(21)}}{2(1)}$$

$$= \frac{10 \pm \sqrt{100-84}}{2}$$

$$= \frac{10 \pm \sqrt{16}}{2}$$

$$= \frac{10 \pm 4}{2} \Rightarrow x = 3 \text{ or } x = 7$$

So, $x^2-10x+21 = (x-3)(x-7)$.

Ex: x^2+1 is irreducible, as $B^2-4AC = 0^2-4(1)(1) = -4 < 0$.

$\Rightarrow$ Can't be factored.

Special Factorizations

- Difference of squares: $x^2-y^2=(x-y)(x+y)$
- Perfect square: $x^2+2xy+y^2=(x+y)^2$
- Difference of cubes: $x^3-y^3=(x-y)(x^2+xy+y^2)$
- Sum of cubes: $x^3+y^3=(x+y)(x^2-xy+y^2)$

Ex: $8x^3 - 1$ is a difference of cubes:

$$8x^3 - 1 = (2x)^3 - (1)^3$$
$$= (2x-1)(4x^2+2x+1)$$

Can we factor $4x^2+2x+1$?

The roots are $x = \dfrac{-2 \pm \sqrt{4-4(4)(1)}}{2(4)}$

$$= \frac{-2 \pm \sqrt{-12}}{8}$$ negative!

Since $B^2-4AC < 0$, $4x^2+2x+1$ is irreducible.

Therefore, $\boxed{8x^3-1 = (2x-1)(4x^2+2x+1)}$

§2.3 - Rational Expressions

Rational expression = quotient of 2 polynomials $\dfrac{p(x)}{q(x)}$ with $q(x) \neq 0$.

Ex: $\dfrac{-5}{x+3}$ (valid for $x \neq -3$), $\dfrac{2x-1}{x^3+x+7}$ (valid when $x^3+x+7 \neq 0$)

We can add rational expressions like we add fractions:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$$

Ex: $\frac{x+2}{x+1} + \frac{5x}{x-1} = \frac{(x+2)(x-1) + 5x(x+1)}{(x+1)(x-1)}$

$= \frac{(x^2+x-2)+(5x^2+5x)}{(x^2-1)}$

$= \boxed{\frac{6x^2+6x-2}{x^2-1}}.$

We can also multiply them like fractions...

$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$

Ex: $\frac{x^3+1}{2x-1} \cdot \frac{7}{x+1} = \frac{7(x^3+1)}{(2x-1)(x+1)} = \boxed{\frac{7x^3+7}{2x^2+x-1}}$

And divide like fractions: $\frac{(a/b)}{(c/d)} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$

Ex: $\frac{3x+1}{x} \Big/ \frac{x+1}{x+2} = \frac{3x+1}{x} \cdot \frac{x+2}{x+1} = \boxed{\frac{3x^2+7x+2}{x^2+x}}$

Sometimes rational expressions can be simplified by factoring the numerator and denominator, and cancelling common terms:

Ex: $\frac{x^2+5x+6}{x^2+x-2} = \frac{(x+3)\cancel{(x+2)}}{\cancel{(x+2)}(x-1)} = \boxed{\frac{x+3}{x-1}}$

§2.4 - Equations (solving for x)

(1) Linear Equations: $3x+2=8 \Rightarrow 3x=6$
$\Rightarrow x = 6/3 = \boxed{2}$

(2) Quadratic Equations

Ex: $x^2+5x+6=0 \Rightarrow \underbrace{(x+2)}\underbrace{(x+3)}=0$

One must be 0

$\Rightarrow \boxed{x=-2}$ or $\boxed{x=-3}$

(Alternatively, use quadratic formula)

Ex: $2x^2+2x+2 = x^2+1 \Rightarrow x^2+2x+1=0$
$\Rightarrow (x+1)^2=0$
$\Rightarrow \boxed{x=-1}$

(3) Rational Equations.

Solve by moving all terms to one side, combining into one rational expression and setting numerator $=0$.

Ex: $\frac{x}{3} = \frac{2x}{x+1} \Rightarrow \frac{x}{3} - \frac{2x}{x+1} = 0$

$$\Rightarrow \frac{x(x+1)-3(2x)}{3(x+1)} = 0$$

$$\Rightarrow \frac{x^2-5x}{3(x+1)} = 0 \Rightarrow x^2-5x=0 \Rightarrow \boxed{x=0,5}$$

§2.5 - Inequalities

<u>Note:</u> Adding and subtracting do not change the direction of the inequality sign, but the sign is <u>reversed</u> when multiplying or dividing by a <u>negative</u> number.

<u>Ex:</u> $7 - x \leq 2x + 5 \quad \Rightarrow \quad -3x \leq -2$

$\Rightarrow \quad x \geq \frac{-2}{-3} = \frac{2}{3}$

sign reversed when dividing by -3.

We can write the solution as

- $x \geq 2/3$
- $x \in [2/3, \infty)$ round means point is excluded

"belongs to" square means point is included (Always round for $\pm\infty$)

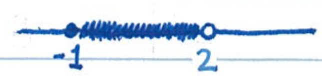
or

-  (number line)

<u>Ex:</u> $-1 \leq 2x + 1 < 5$ (We can solve both inequalities at once!)

$\Rightarrow \quad -2 \leq 2x < 4$ (subtract 1 on all sides)

$\Rightarrow \quad \boxed{-1 \leq x < 2}$ (divide by 2 on all sides)

We could also write $x \in [-1, 2)$ or 

Rational Inequalities

Ex 1. $\dfrac{x+1}{x-2} > 2$ Ex 2. $x^2 - 2x - 8 \geq 0$

- Move all terms to one side and write as a single rational expression.
- Find all values where numerator or denominator is 0
- Plot these on a number line and check points between these values. Include the interval if the inequality is true.

Let's try this approach in Ex 1 above.

$$\frac{x+1}{x-2} > 2 \Rightarrow \frac{x+1}{x-2} - 2 > 0 \Rightarrow \frac{x+1-2(x-2)}{x-2} > 0.$$

Simplify to get $\dfrac{5-x}{x-2} > 0$.

Numerator is 0 at $x=5$, denominator is 0 at $x=2$.

Test

By testing points in each interval (say $x=0$, $x=3$, $x=6$) we see the inequality only holds for $\boxed{x \in (2,5)}$.
(use round brackets to exclude $x=2$ and $x=5$).

Note: The expression in Ex 2. is rational!
(The denominator is 1 so we'll ignore it.)

The numerator is $x^2 - 2x - 8 = (x-4)(x+2)$, so it is 0 when $x = -2$ or $x = 4$.

Test

Test points in each interval (say $x = -3$, $x = 0$, $x = 5$) to see that the inequality holds for $\boxed{x \in (-\infty, -2) \cup (4, \infty)}$

§ 2.6 - Exponents

Recall that for $n \geq 1$, $x^n = \underbrace{x \cdot x \cdot \cdots \cdot x}_{n \text{ times}}$

Also $x^0 = 1$, $x^{-1} = \frac{1}{x}$, and $x^{-n} = \frac{1}{x^n}$.

Useful Properties

$a^m \cdot a^n = a^{m+n}$ $\qquad (a \cdot b)^m = a^m \cdot b^m$

$a^m / a^n = a^{m-n}$ $\qquad (a^m)^n = a^{m \cdot n}$

Note: $a^m + a^n$ does not simplify to anything nice...

Ex: Simplify the following expressions.

(1) $5^3 \cdot 5^{11} = 5^{3+11} = \boxed{5^{14}}$

(2) $(2x^3)^4 = 2^4 (x^3)^4 = \boxed{2^4 x^{12}}$

(3) $\left(\dfrac{x^2}{y^5}\right)^7 = \dfrac{(x^2)^7}{(y^5)^7} = \boxed{\dfrac{x^{14}}{y^{35}}}$

(4) $\dfrac{a^{-3} b^5}{a^4 b^{-7}} = \dfrac{b^5 b^7}{a^4 a^3} = \boxed{\dfrac{b^{12}}{a^7}}$

What about when we have fractions as exponents?

Recall that $x^{1/n} = \sqrt[n]{x}$ (the n^{th} root of x)

and $a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$.

Ex:
- $9^{1/2} = \sqrt{9} = \boxed{3}$
- $27^{1/3} = \sqrt[3]{27} = \boxed{3}$
- $4^{3/2} = (4^{1/2})^3 = (\sqrt{4})^3 = 2^3 = \boxed{8}$

(Alternatively, $4^{3/2} = (4^3)^{1/2} = \sqrt{64} = \boxed{8}$)

§2.7 - Radicals (Roots)

Radical = n^{th} root : $\sqrt[n]{a} = a^{1/n}$ for natural number n.

Note: If n is an even natural number, $\sqrt[n]{a}$ exists only for $a \geq 0$.

Properties: • $\left(\sqrt[n]{a}\right)^n = a$ (if it exists)

• $\sqrt[n]{a/b} = \sqrt[n]{a} / \sqrt[n]{b}$

• $\sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b}$

• $\sqrt[m]{\sqrt[n]{a}} = \sqrt[m \cdot n]{a}$

• $\sqrt[n]{a^n} = \begin{cases} |a| & \text{if } n \text{ is even} \\ a & \text{if } n \text{ is odd.} \end{cases}$

Ex: $\sqrt[4]{16/81} = \sqrt[4]{16} / \sqrt[4]{81} = 2/3$

Ex: $\sqrt[3]{-27} = -3$

Ex: $\sqrt{-36}$ does not exist

Ex: $\sqrt{(-3)^2} = |-3| = 3$, $\sqrt[3]{10^3} = 10$.

Simplify the following:

(1) $\sqrt[3]{24}$ (2) $\sqrt{x^3y^2}$ assuming $x \geq 0, y \geq 0$.

For (1), we have $24 = 8 \cdot 3 = 2^3 \cdot 3$,

so $\sqrt[3]{24} = \sqrt[3]{2^3 \cdot 3}$

$= \sqrt[3]{2^3}\,\sqrt[3]{3}$

$= \boxed{2\sqrt[3]{3}}$

For (2), we have $\sqrt{x^3y^2} = \sqrt{x^2 \cdot x \cdot y^2} = \boxed{xy\sqrt{x}}$

Note: When simplifying a quotient, it helps to rationalize the denominator.

Ex: $\dfrac{1}{1+\sqrt{3}} = \dfrac{1}{1+\sqrt{3}} \cdot \dfrac{1-\sqrt{3}}{1-\sqrt{3}} = \dfrac{1-\sqrt{3}}{1-3} = \boxed{\dfrac{\sqrt{3}-1}{2}}$

Caution: $\sqrt{a^2+b^2} \neq \sqrt{a^2} + \sqrt{b^2}$!!

This expression doesn't simplify nicely.