Applications of Integration.

It turns out that integrals have lots of applications. Some areas where integrals show up include probability, physics, finance, and economics.

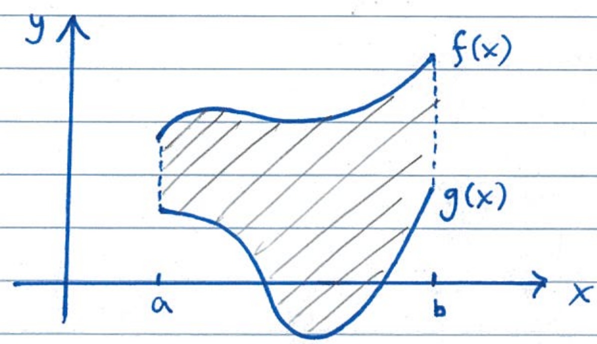
Here we study two applications:

1. Centre of mass
2. Work.

Centre of Mass (also called a centroid)

A region's centre of mass is a point $(\bar{x}, \bar{y})$ so that if the region is suspended from that point, it won't tilt at all.

Assume the region has uniform density ρ, and that it is between $f(x)$ and $g(x)$ from $x = a$ to $x = b$.



We'll need 3 things...

The region's mass is $m = \text{density} \cdot \text{area}$:

$$\boxed{m = \rho \cdot \int_a^b f(x) - g(x) \, dx.}$$

The region's x-moment is

$$\boxed{m_x = \rho \cdot \int_a^b \frac{1}{2}\left([f(x)]^2 - [g(x)]^2\right) dx}$$

and its y-moment is

$$\boxed{m_y = \rho \cdot \int_a^b x\left(f(x) - g(x)\right) dx.}$$

(These measure the region's tendency to rotate in the x or y direction)

Using these values, we can compute the centre of mass $(\bar{x}, \bar{y})$:

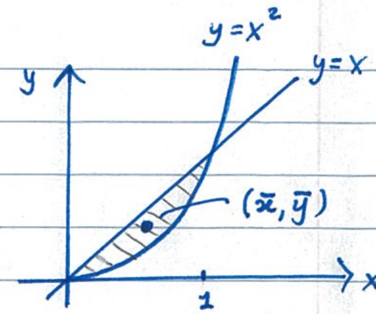
$$\bar{x} = \frac{m_y}{m} = \frac{\int_a^b x\left(f(x) - g(x)\right) dx}{\int_a^b f(x) - g(x)\, dx}$$

$$\bar{y} = \frac{m_x}{m} = \frac{\int_a^b \frac{1}{2}\left([f(x)]^2 - [g(x)]^2\right) dx}{\int_a^b f(x) - g(x)\, dx}$$

Remark: The density ρ cancelled out!

Ex: Find the centre of mass for the region between $y = x$ and $y = x^2$, from $x = 0$ to $x = 1$.

<u>Solution</u>: Here's the picture



To find $(\bar{x}, \bar{y})$, we need m, m_x, and m_y.

(We'll ignore density since we know this will cancel out!)

$$m = \int_0^1 x - x^2 \, dx = \frac{x^2}{2} - \frac{x^3}{3} \Big|_0^1$$

$$= \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{0}{2} - \frac{0}{3}\right) = \boxed{\frac{1}{6}}$$

$$m_x = \int_0^1 \frac{1}{2}\left((x)^2 - (x^2)^2\right) dx$$

$$= \int_0^1 \frac{1}{2}(x^2 - x^4)\, dx = \frac{1}{2}\left(\frac{x^3}{3} - \frac{x^5}{5}\right)\Big|_0^1$$

$$= \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right) = \boxed{\frac{1}{15}}$$

$$m_y = \int_0^1 x(x - x^2)\, dx$$

$$= \int_0^1 x^2 - x^3 \, dx = \frac{x^3}{3} - \frac{x^4}{4}\Big|_0^1$$

$$= \left(\frac{1}{3} - \frac{1}{4}\right) = \boxed{\frac{1}{12}}$$

So $(\bar{x}, \bar{y}) = \left(\frac{m_y}{m}, \frac{m_x}{m}\right) = \left(\frac{1/12}{1/6}, \frac{1/15}{1/6}\right) = \boxed{\left(\frac{1}{2}, \frac{2}{5}\right)}$

Work

Q: If an object is moved a distance d in a straight line with force F, how much work is done on the object?

A: Work = Force $\cdot$ distance, so $W = F \cdot d$.
If it is moved from $x=a$ to $x=b$, $W = F \cdot (b-a)$.

However, F may not be a constant force. Instead, it could be a function of x, $F(x)$.

In this case...

$$\boxed{W = \int_a^b F(x)\, dx}$$

Ex: A 50m cable hangs from a building with a total mass of 200 Kg. How much work is required to lift the cable to the top of the building?

Solution:

Gravity is the force acting on the cable.

Recall that force = mass $\cdot$ acceleration, and acceleration from gravity is $9.81\ m/s^2$

What's the mass? Well... each metre of cable has a mass of $\frac{200\text{ Kg}}{50\text{ m}} = 4\text{ Kg/m}$.

So after x metres have been lifted (and $(50-x)$ metres remain), the mass is $4(50-x)$ Kg.

This means that

$$F(x) = \text{mass} \cdot \text{acceleration}$$
$$= 4(50-x) \cdot 9.81.$$

Hence, $W = \int_0^{50} 4(50-x) \cdot 9.81 \, dx$

$$= 4 \cdot 9.81 \left(50x - \frac{x^2}{2}\right)\Big|_0^{50}$$

$$= 39.24 \left(50(50) - \frac{(50)^2}{2}\right)$$

$$= 39.24 \left(2500 - 2500/2\right)$$

$$= 39.24 \left(\frac{2500}{2}\right) = \boxed{49050 \text{ J (Joules)}}.$$