§9.1 – Numerical Integration

Some functions out in the wild have no antiderivative

Ex: $f(x) = e^{x^2}$ has no antiderivative -- WEIRD!

This means that FTC cannot be used to compute $\int_a^b e^{x^2}\,dx$, so we need to use estimates

1. Left endpoints
2. Right endpoints
3. Midpoints
4. Trapezoids.

If we're going to use estimates, we should ask the following 2 questions:

(1) How good is the estimate?

(2) How many points are required to get a good estimate?

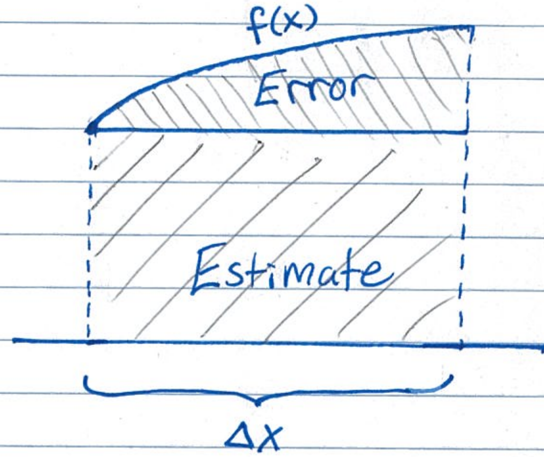
How Good is an Estimate?

Analysis: Left endpoint method.

The biggest factors affecting our estimate are

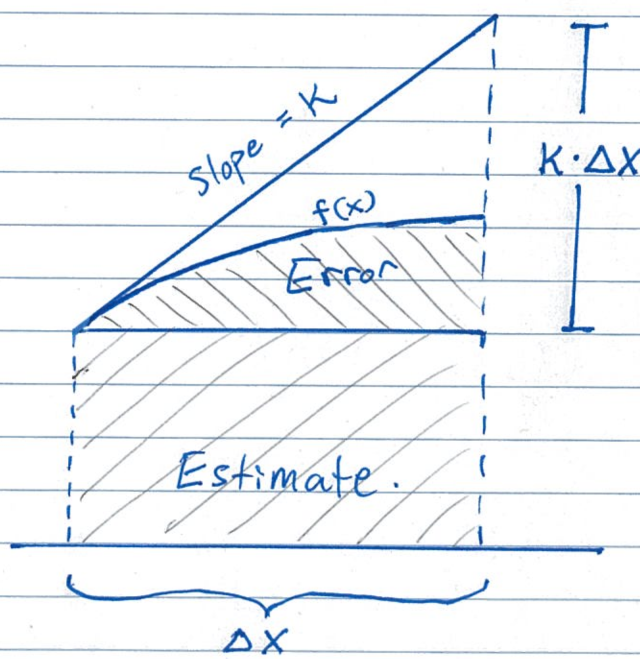
1. Number of rectangles used (n), and
2. the shape of f(x).
   (flatter function, better estimate)

Let's look at just 1 rectangle to start:



Let $K = \max|f'(x)|$ for $x \in [a,b]$, so $f(x)$ has slope at most K.

By using a line with slope K, we can get an easy upper bound on the error:



Error $\leq$ Area of triangle $= \frac{1}{2} \cdot$ base $\cdot$ height

$$= \frac{1}{2} \cdot \Delta x \cdot (K \cdot \Delta x) = \frac{K(\Delta x)^2}{2}$$

By adding up over all n rectangles, we get

$$\text{Error} \leq \frac{K(\Delta x)^2}{2} + \frac{K(\Delta x)^2}{2} + \cdots + \frac{K(\Delta x)^2}{2}$$

(n times)

$$= \frac{n \cdot K(\Delta x)^2}{2}$$

But $\Delta x = \dfrac{b-a}{n}$, so...

$$\text{Error} \leq n \cdot \frac{K}{2}\left(\frac{b-a}{n}\right)^2 = \frac{K(b-a)^2}{2n}$$

Similar constructions for the other methods give us the following theorem:

<u>Theorem</u>: Let $K_1 = \max|f'(x)|$ for $x \in [a,b]$,
and $K_2 = \max|f''(x)|$ for $x \in [a,b]$.

Then,

Method	Error
Left endpoints	$\leq \dfrac{K_1(b-a)^2}{2n}$
Right endpoints	$\leq \dfrac{K_1(b-a)^2}{2n}$
Midpoints	$\leq \dfrac{K_2(b-a)^3}{24n^2}$
Trapezoids	$\leq \dfrac{K_2(b-a)^3}{12n^2}$

Ex: Find an upper bound on the error of estimating $\int_0^2 \sin x \, dx$ with 10 data points using each of the four methods above.

Solution:

If $f(x) = \sin x$, then $f'(x) = \cos x$

$f''(x) = -\sin x$.

So $K_1 = \max|\cos x|$ for $x \in [0,2]$

$K_2 = \max|-\sin x|$ for $x \in [0,2]$.

Since $|\cos x| \le 1$ and $|\sin x| \le 1$ for all x, we can take $K_1 = 1$ and $K_2 = 1$.

1. Left endpoints: error $\le \dfrac{1\cdot(2-0)^2}{2(10)} = \boxed{\dfrac{1}{5}}$

2. Right endpoints: error $\le \boxed{\dfrac{1}{5}}$ (same as left endpoints)

3. Midpoints: error $\le \dfrac{1\cdot(2-0)^3}{24(10)^2} = \boxed{\dfrac{1}{300}}$

4. Trapezoids: error $\le \dfrac{1\cdot(2-0)^3}{12(10)^2} = \boxed{\dfrac{1}{150}}$

Ex: How many data points would be required to estimate $\int_0^1 e^{x^2} \, dx$ to an accuracy of $\dfrac{1}{1000}$ using the right endpoint method? Midpoint method?

Solution: Since we'll be using our error estimates, we should first find K_1 and K_2.

If $f(x) = e^{x^2}$, then

$$f'(x) = e^{x^2}(x^2)' = 2x \cdot e^{x^2}$$

$$f''(x) = 2e^{x^2} + 2x(2xe^{x^2}) = (2+4x^2)e^{x^2}.$$

Notice that both $f'(x)$ and $f''(x)$ are ≥ 0 and increasing on $[0,1]$, so

$$K_1 = \max |2x \cdot e^{x^2}| = 2(1)e^{(1)^2} = 2e$$

$$K_2 = \max |(2+4x^2)e^{x^2}| = (2+4(1)^2)e^{(1)^2} = 6e.$$

First, let's use right endpoints. We want n so that error $\leq \frac{1}{1000}$.

We know error $< \frac{K_1(b-a)^2}{2n} = \frac{2e(1-0)^2}{2n} = \frac{e}{n}$

and $\frac{e}{n} \leq \frac{1}{1000} \Rightarrow 1000e \leq n$

$$\Rightarrow n \geq 2718.28\ldots$$

$$\Rightarrow \boxed{n \geq 2719}$$

(ew!)

For midpoints, we have error $< \frac{K_2(b-a)^3}{24n^2} = \frac{6e}{24n^2}$

So $\frac{6e}{24n^2} \leq \frac{1}{1000} \Rightarrow 250e \leq n^2$

$$\Rightarrow n \geq \sqrt{250e} \approx 26.06$$

$$\Rightarrow \boxed{n \geq 27}$$

(better!)