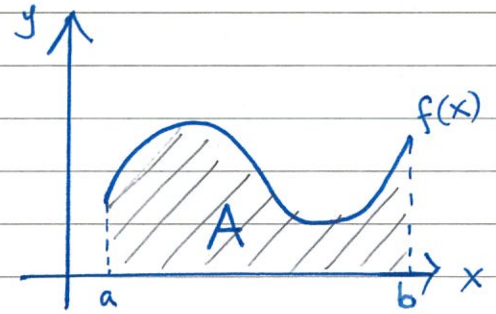
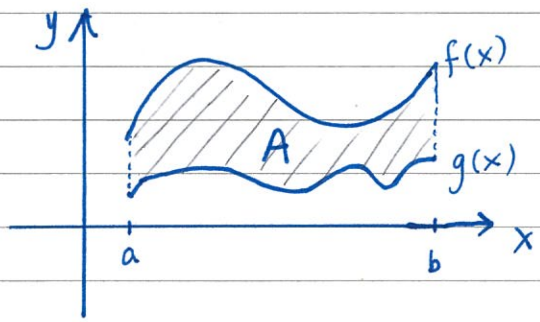
§8.6 – Areas Between Curves

We know how to find the area between $f(x)$ and the x-axis:



$$A = \int_a^b f(x)\,dx$$

But what about the area between $f(x)$ and $g(x)$?



$A = ???$

This area A is just the area under $f(x)$, minus the area under $g(x)$.

In general, it's the area under the upper function, minus the area under the lower function.

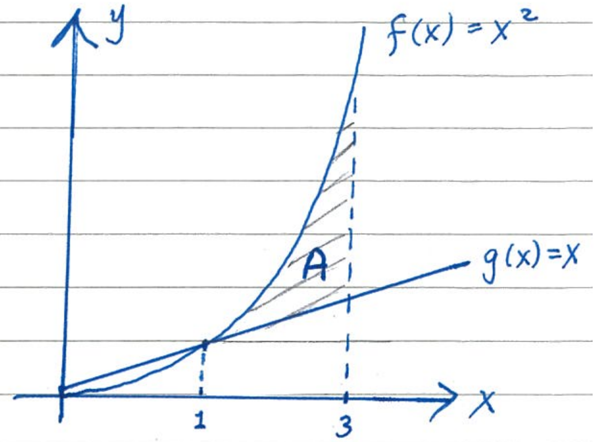
So... if $f(x) \geq g(x)$ on $[a,b]$, then the area between $f(x)$ and $g(x)$ from $x=a$ to $x=b$ is

$$A = \int_a^b f(x) - g(x)\,dx$$

Ex: Find the area between $f(x) = x^2$ and $g(x) = x$ from $x=1$ to $x=3$.

Solution:

Looks like $f(x) = x^2$ is the upper function on $[1,3]$, and $g(x) = x$ is the lower function



So, $A = \int_1^3 x^2 - x \, dx$

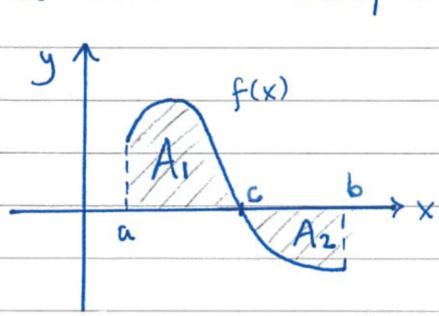
$$= \frac{x^3}{3} - \frac{x^2}{2} \Big|_1^3 = \left(\frac{3^3}{3} - \frac{3^2}{2}\right) - \left(\frac{1^3}{3} - \frac{1^2}{2}\right)$$

$$= \left(\frac{27}{3} - \frac{9}{2}\right) - \left(\frac{1}{3} - \frac{1}{2}\right)$$

$$= \frac{26}{3} - \frac{8}{2} = \boxed{\frac{14}{3}}$$

Remark: If you mix up the upper and lower functions, your answer will be negative. Go back and check your work!

In the last section we saw that if $f(x)$ is sometimes above the x-axis and sometimes below, we need to compute 2 integrals:

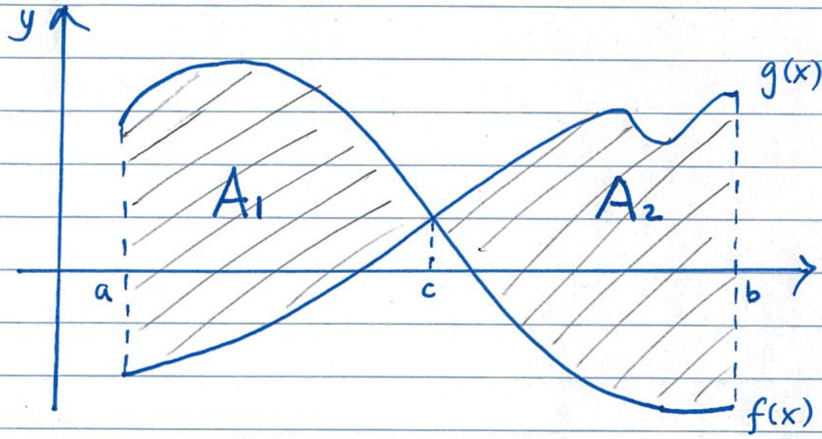


$A = A_1 + A_2$

$A_1 = \int_a^c f(x) \, dx \qquad A_2 = -\int_c^b f(x) \, dx$

Something similar happens here!

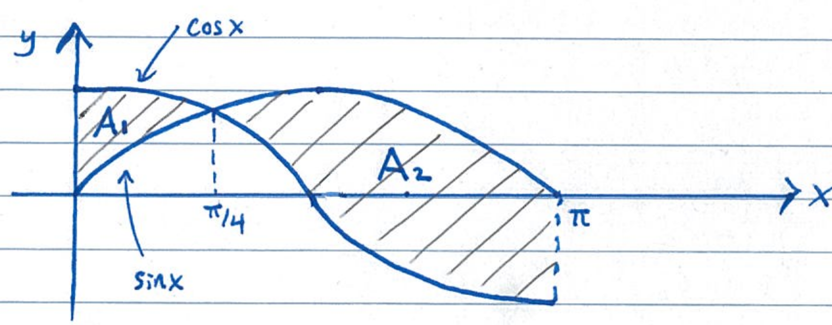
If the upper and lower functions change over $[a, b]$, we will have to compute more than one integral.



$$A = A_1 + A_2 \qquad A_1 = \int_a^c f(x) - g(x)\,dx \qquad A_2 = \int_c^b g(x) - f(x)\,dx$$

Ex: Find the area between $f(x) = \cos x$ and $g(x) = \sin x$ from $x = 0$ to $x = \pi$.

Solution:



It looks like $\cos x$ starts on top, and then after some point $\sin x$ is on top.

what point? Well... $\cos x = \sin x$ when $x = \frac{\pi}{4}$.

This is from our unit circle.

So $A = A_1 + A_2$

$$= \int_0^{\pi/4} \cos x - \sin x \, dx + \int_{\pi/4}^{\pi} \sin x - \cos x \, dx$$

$$= (\sin x + \cos x)\Big|_0^{\pi/4} + (-\cos x - \sin x)\Big|_{\pi/4}^{\pi}$$

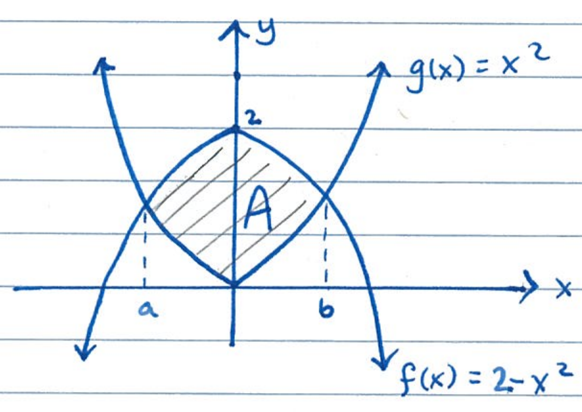
$$= \left(\sin\left(\tfrac{\pi}{4}\right) + \cos\left(\tfrac{\pi}{4}\right)\right) - \left(\sin(0) + \cos(0)\right)$$
$$+ \left(-\cos(\pi) - \sin(\pi)\right) - \left(-\cos\left(\tfrac{\pi}{4}\right) - \sin\left(\tfrac{\pi}{4}\right)\right)$$

$$= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - (0 + 1) + (-(-1) - 0) - \left(\frac{-\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right)$$

$$= \sqrt{2} - \cancel{1} + \cancel{1} + \sqrt{2} \quad = \boxed{2\sqrt{2}}$$

Ex: Find the area enclosed by $f(x) = 2 - x^2$ and $g(x) = x^2$.

Solution:



Looks like $f(x) = 2 - x^2$ is the upper function.
But what are a and b ??

They are the points where f(x) and g(x) intersect.

$f(x) = g(x) \Rightarrow 2 - x^2 = x^2$

$\Rightarrow 2 = 2x^2$

$\Rightarrow 1 = x^2$

$\Rightarrow x = \pm 1$

So $$A = \int_{-1}^{1} (2-x^2) - x^2 \, dx$$

$$= \int_{-1}^{1} 2 - 2x^2 \, dx$$

$$= 2x - \frac{2x^3}{3} \Big|_{-1}^{1}$$

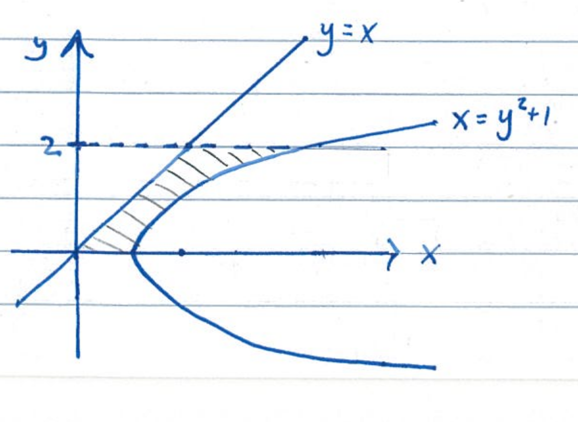
$$= \left(2(1) - \frac{2(1)^3}{3}\right) - \left(2(-1) - \frac{2(-1)^3}{3}\right)$$

$$= 2 - \frac{2}{3} + 2 - \frac{2}{3} = 4 - \frac{4}{3} = \boxed{\frac{8}{3}}$$

We can also use these ideas to compute areas when x is a function of y. Instead of "upper minus lower" we use "rightmost minus leftmost".

Ex: Find the area between $x = y^2+1$ and $y = x$ from $y=0$ to $y=2$.

Solution:



$$A = \int_0^2 \underbrace{(y^2+1)}_{\text{rightmost}} - \underbrace{y}_{\text{leftmost}} \, dy$$

$$= \frac{y^3}{3} + y - \frac{y^2}{2} \Big|_0^2$$

$$= \frac{8}{3} + 2 - \frac{4}{2} = \boxed{\frac{8}{3}}$$