§8.4 - The Fundamental Theorem of Calculus

Here we introduce a powerful theorem that allows us to compute $\int_a^b f(x)\,dx$ without the limits in §8.3.

The Fundamental Theorem of Calculus (FTC)

Suppose $f(x)$ is continuous on $[a, b]$ and let $F(x)$ be any antiderivative of $f(x)$ (so $F'(x) = f(x)$). Then

$$\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_a^b$$

new notation

So, rather than computing a horrendous limit, we can find the exact area under $f(x)$ using antiderivatives!!

EX: Find the area under $f(x) = x^2 + x$ from 0 to 4.

Solution:

$$\int_0^4 x^2 + x\,dx = \frac{x^3}{3} + \frac{x^2}{2} + C\Big|_0^4$$

$$= \left(\frac{4^3}{3} + \frac{4^2}{2} + \cancel{C}\right) - \left(\cancel{\frac{0^3}{3}} + \cancel{\frac{0^2}{2}} + \cancel{C}\right) = \boxed{\frac{88}{3}}$$

Notice that the "+C" terms just cancelled out. This will always happen when computing $\int_a^b f(x)\,dx$, so...

When computing $\int_a^b f(x)dx$, we don't need +C !

Recap:

$\int f(x)\,dx$ = indefinite integral (general antiderivative)

$= F(x) + C$

$\int_a^b f(x)\,dx$ = definite integral (area under $f(x)$ from a to b)

$= F(b) - F(a)$

Properties of the Definite Integral

1. $\int_a^a f(x)\,dx = 0$

2. $\int_a^b k \cdot f(x)\,dx = k \cdot \int_a^b f(x)\,dx$

3. $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

4. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

5. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ for any $c \in \mathbb{R}$

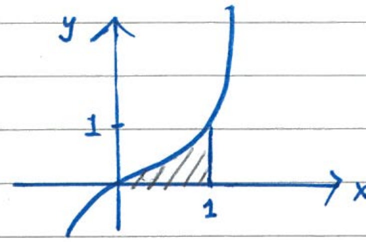
Important remark: $\int_a^b f(x)\,dx$ is actually the signed area "under" $f(x)$

So, if $f(x)$ is above the x-axis, then it's positive and if $f(x)$ is below the x-axis, then it's negative.

(all of our examples so far have been above the x-axis)

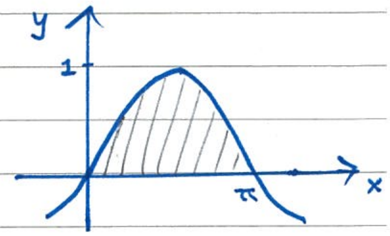
Ex:

1. $\int_0^1 x^3\,dx = \left.\frac{x^4}{4}\right|_0^1$



$= \left(\frac{1^4}{4}\right) - \left(\frac{0^4}{4}\right) = \boxed{\frac{1}{4}}$

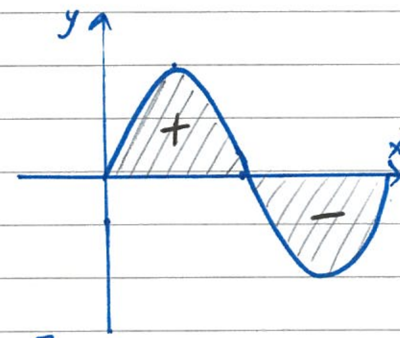
2. $\int_0^\pi \sin x\,dx = -\cos x\Big|_0^\pi$



$= (-\cos\pi) - (-\cos(0))$

$= 1 - (-1) = \boxed{2}$

3. $\int_0^{2\pi} \sin x\,dx = -\cos x\Big|_0^{2\pi}$

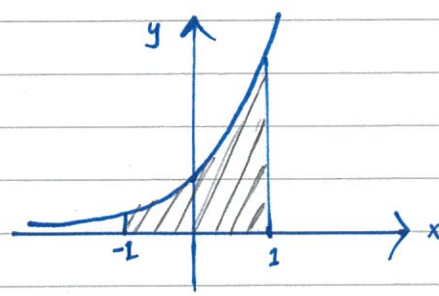


$= (-\cos(2\pi)) - (-\cos(0))$

$= (-1) - (-1) = \boxed{0}$

Areas cancel!

4. $\int_{-1}^1 e^{2t}\,dt = \left.\frac{e^{2t}}{2}\right|_{-1}^1$



$= \boxed{\frac{e^2}{2} - \frac{e^{-2}}{2}}$

Important note: If you make a substitution, don't forget to change a and b!!

Ex: Calculate $\int_1^3 x\cos(x^2+1)\,dx$

Solution: Let $u = x^2+1$

So $du = 2x\,dx \Rightarrow dx = \frac{du}{2x}$

When $x=1$, $u = (1)^2+1 = 2$

When $x=3$, $u = (3)^2+1 = 10$

So, $\int_1^3 x\cos(x^2+1)\,dx = \int_2^{10} x\cos(u)\frac{du}{2x}$

$= \frac{1}{2}\int_2^{10} \cos(u)\,du$

$= \frac{1}{2}\sin(u)\Big|_2^{10} = \boxed{\frac{1}{2}[\sin(10)-\sin(2)]}$

Exercise: Calculate $\int_1^e \frac{3}{x(1+\ln x)}\,dx$

Sometimes we may want the area between $f(x)$ and the x-axis. We do this in 3 steps.

1. Find x-intercepts
2. Integrate $f(x)$ between each pair of intercepts.
3. Add together these areas in absolute value.

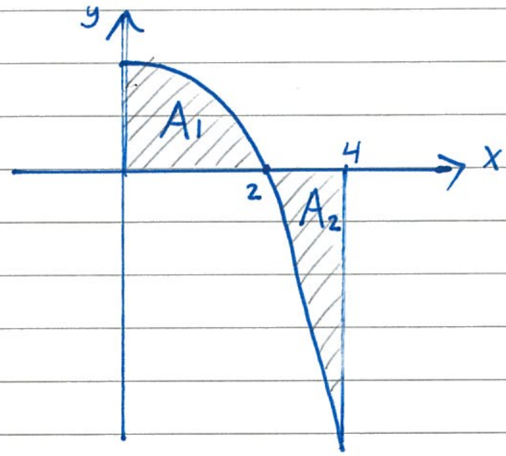
Ex: Find the area between $f(x) = 4-x^2$ and the x-axis in $[0,4]$.

Solution:

What are the x-intercepts in $[0,4]$?

$4-x^2=0 \Rightarrow x^2=4$

$\Rightarrow x=\pm 2$ (only $+2$ is in $[0,4]$)



$$A_1=\int_0^2 4-x^2\,dx = 4x-\frac{x^3}{3}\Big|_0^2$$

$$=\left(4(2)-\frac{(2)^3}{3}\right)-\left(4(0)-\frac{(0)^3}{3}\right)$$

$$=\frac{16}{3}$$

$$A_2=\int_2^4 4-x^2\,dx = 4x-\frac{x^3}{3}\Big|_2^4$$

$$=\left(4(4)-\frac{4^3}{3}\right)-\left(4(2)-\frac{2^3}{3}\right)$$

$$=-\frac{32}{3}$$

So Area $=\left|\frac{16}{3}\right|+\left|\frac{-32}{3}\right| = \frac{16}{3}+\frac{32}{3}=\frac{48}{3}=\boxed{16}$